Math 262 — 13 December 2022
End-of-Semester Review Problems

1. Let $\phi(x) = \alpha f(x) + \beta g(x)$. Under what conditions on the constants α and β will the $\phi(x)$ be a pdf for all possible pdfs $f(x)$ and $g(x)$?

Since f and g are pdfs:

$$f(x) \geq 0, \quad g(x) \geq 0, \quad \int_{-\infty}^{\infty} f(x)\,dx = 1, \quad \text{and} \quad \int_{-\infty}^{\infty} g(x)\,dx = 1$$

If $\alpha f(x) + \beta g(x)$ is a pdf, for all possible pdfs $f(x)$ and $g(x)$, it must be that $\alpha \geq 0$, $\beta \geq 0$, and:

$$1 = \int_{-\infty}^{\infty} (\alpha f(x) + \beta g(x))\,dx = \alpha \int_{-\infty}^{\infty} f(x)\,dx + \beta \int_{-\infty}^{\infty} g(x)\,dx = \alpha + \beta$$

So $\alpha + \beta = 1$.

2. Let $X \sim \text{Exp}(\lambda)$, $0 \leq s$ and $0 \leq t$. Since X is memoryless, is it true that $(X > s + t)$ and $(X > t)$ are independent events?

Memoryless Property: $P(X > s+t \mid X > t) = P(X > s)$

Since $P(X > s) \neq P(X > s+t)$, we have

$$P(X > s+t \mid X > t) \neq P(X > s+t),$$

so the events $X > s+t$ and $X > t$ are not independent.

3. Let X and Y be iid exponential random variables with parameter λ. Let (R, Θ) denote the polar coordinates of (X, Y). What is the joint density of R and Θ?

The joint density of X and Y is $f(x,y) = \lambda^2 e^{-\lambda(x+y)}$.

Since X and Y are nonnegative, (X, Y) is in the first quadrant, so $0 \leq R$ and $0 \leq \Theta \leq \frac{\pi}{2}$.

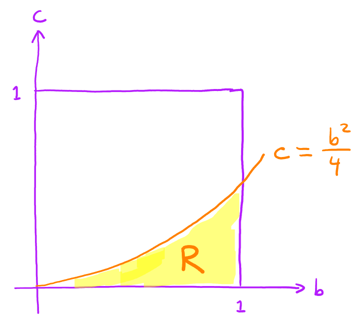
As in the video on joint transformations, the transformation theorem gives the joint density of R and Θ:

$$g(r, \theta) = f(r \cdot \cos\theta,\ r \cdot \sin\theta) \cdot r$$
$$= \lambda^2 e^{-\lambda r(\cos\theta + \sin\theta)} \cdot r \qquad \text{for } 0 \leq r,\ 0 \leq \theta \leq \tfrac{\pi}{2}.$$

4. Suppose B and C are iid Unif[0,1]. Find the probability that the roots of the equation $x^2 + Bx + C = 0$ are real.

The roots are $x = \dfrac{-B \pm \sqrt{B^2 - 4C}}{2}$, which are real iff $B^2 - 4C \geq 0$,

or equivalently, $C \leq \dfrac{B^2}{4}$.



Then: $P(\text{real roots}) = P\left(C \leq \frac{B^2}{4}\right)$

$$= \iint_R 1 \, dA = \text{Area}(R)$$

$$= \int_0^1 \frac{b^2}{4} \, db = \frac{1}{12}$$

5. Alina makes 100 flips of a fair coin, and Dennis makes 99 flips of a fair coin. What is the probability that Alina gets *more* heads than Dennis?

Let A be the number of heads that Alina gets in the first 99 flips.

Let D be the number of heads that Dennis gets in 99 flips.

Let $q = P(A > D)$. By symmetry, $q = P(D > A)$ also.

Then $P(A = D) = 1 - 2q$.

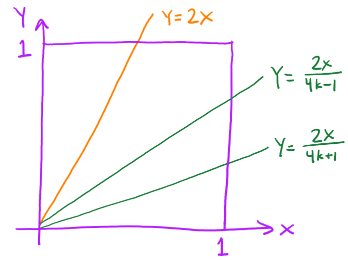
After 99 coin flips, Alina still gets one more coin flip.

Thus, Alina gets more heads than Dennis if $A > D$ or ($A = D$ and Alina's last flip is heads).

So: $P(\text{Alina gets more heads than Dennis}) = q + (1-2q) \cdot \frac{1}{2}$

$$= q + \frac{1}{2} - q = \boxed{\frac{1}{2}}$$

6. X and Y are iid Unif[0,1]. What is the probability that the closest integer to $\frac{X}{Y}$ is even?



(a) $\frac{X}{Y}$ is closest to zero if $\frac{X}{Y} < \frac{1}{2}$, or $2X < Y$.

$$P(2X < Y) = \frac{1}{4}$$

(b) If k is a positive integer, then $\frac{X}{Y}$ is closest to $2k$ if $\frac{4k-1}{2} < \frac{X}{Y} < \frac{4k+1}{2}$.

$$P\left(\frac{4k-1}{2} < \frac{X}{Y} < \frac{4k+1}{2}\right) = P\left(\frac{2X}{4k+1} < Y < \frac{2X}{4k-1}\right) = \frac{1}{4k-1} - \frac{1}{4k+1}$$

Combining parts (a) and (b), we have:

$$P\left(\text{integer closest to } \frac{X}{Y} \text{ is even}\right) = \frac{1}{4} + \sum_{k=1}^{\infty}\left(\frac{1}{4k-1} - \frac{1}{4k+1}\right)$$

← Mathematica can tell you this is $\frac{5-\pi}{4}$.

$$= \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \frac{1}{13} + \cdots$$

This sum is $1 - \frac{\pi}{4}$.

(Write the Taylor series for $\arctan(x)$ at $x = \frac{\pi}{4}$.)

$$= \frac{1}{4} + 1 - \frac{\pi}{4} = \boxed{\frac{5-\pi}{4}}$$