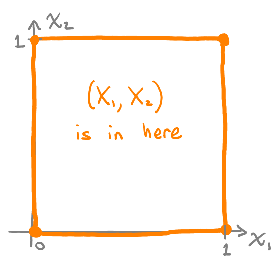
4. Suppose that X_1 and X_2 are iid Unif[0,1]. Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.

(a) Find the region of possible values of the pair (Y_1, Y_2).

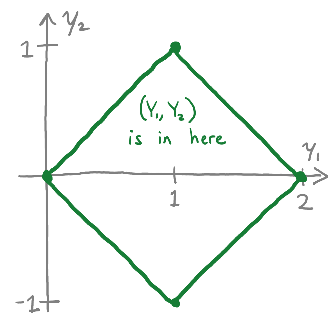
Note that the transformation is linear: $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$

Linear transformations map parallelograms to parallelograms.

The point (X_1, X_2) lies in a square. Let's see where the transformation maps this square:



$(0,0) \mapsto (0,0)$
$(1,0) \mapsto (1,1)$
$(1,1) \mapsto (2,0)$
$(0,1) \mapsto (1,-1)$



(b) Find the inverse transformation functions v_1 and v_2 such that $X_1 = v_1(Y_1, Y_2)$ and $X_2 = v_2(Y_1, Y_2)$.

Invert the transformation matrix: $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

So $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$, which means

$$X_1 = v_1(Y_1, Y_2) = \frac{Y_1 + Y_2}{2} \quad \text{and} \quad X_2 = v_2(Y_1, Y_2) = \frac{Y_1 - Y_2}{2}$$

(c) Use the transformation theorem to find the joint pdf of Y_1 and Y_2.

Jacobian matrix: $M = \begin{bmatrix} \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial y_2} \\ \frac{\partial v_2}{\partial y_1} & \frac{\partial v_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

So: $\det(M) = \frac{1}{2}\left(-\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}\right) = -\frac{1}{2}$

The joint density of Y_1 and Y_2 is:

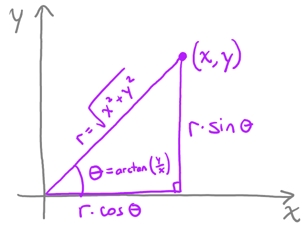
$$g(y_1, y_2) = f\left(v_1(y_1, y_2), v_2(y_1, y_2)\right) \cdot \left|\det(M)\right| = 1 \cdot \left|\tfrac{1}{2}\right| = \tfrac{1}{2}$$

on the region found in part (a).

5. Let (X, Y) be a random point in the plane, where X and Y are independent standard normal random variables. Let (R, Θ) be the polar coordinates of (X, Y). Find the joint density of R and Θ. Then find the marginal densities of R and Θ. What is the probability that the point (X, Y) lies in a circle of radius 1 centered at the origin?

Transformations:

$$(X,Y) \underset{X = R\cdot\cos\theta,\ \ Y = R\cdot\sin\theta}{\overset{R = \sqrt{X^2+Y^2},\ \ \theta = \arctan\left(\frac{X}{Y}\right)}{\rightleftarrows}} (R,\theta)$$



Region: X and Y may be any real numbers, so $R \geq 0$ and $0 \leq \theta \leq 2\pi$.

Joint Density: $N(0,1)$ has pdf $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

X and Y are independent, so $f(x,y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$

$$f(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

From the video for today, the transformation theorem gives:

$$g(r,\theta) = f(r\cdot\cos\theta,\ r\cdot\sin\theta)\cdot r$$

Thus: $$g(r,\theta) = \frac{1}{2\pi} e^{-(r^2\cos^2\theta + r^2\sin^2\theta)/2} \cdot r$$

$$g(r,\theta) = \frac{r}{2\pi} e^{-r^2/2} \quad \text{for } r \geq 0,\ 0 \leq \theta < 2\pi$$

Marginal densities:

$\Theta \sim \text{Unif}[0, 2\pi]$

of θ: $$g_\theta(\theta) = \int_0^\infty \frac{r}{2\pi} e^{-r^2/2}\, dr = \frac{1}{2\pi} e^{-r^2/2} \Big|_{r=0}^{r=\infty} = \boxed{\frac{1}{2\pi} \quad \text{for } 0 \leq \theta \leq 2\pi}$$

of R: $$g_r(\theta) = \int_0^{2\pi} \frac{r}{2\pi} e^{-r^2/2}\, d\theta = \frac{r}{2\pi} e^{-r^2/2}\theta \Big|_{\theta=0}^{\theta=2\pi} = \boxed{r e^{-r^2/2} \quad \text{for } r \geq 0.}$$

R has a Rayleigh distribution

Inside circle:

$$P(R<1) = \int_0^1 r e^{-r^2/2}\, dr = -e^{-r^2/2} \Big|_{r=0}^{r=1} = 1 - e^{-1/2} \approx 0.393$$

1. Let X_1 and X_2 be iid Exp$\left(\frac{1}{10}\right)$. $f(x) = \frac{1}{10} e^{-x/10}$ for $x>0$, $F(x) = 1 - e^{-x/10}$ for $x > 0$

(a) What is the pdf of $Y_1 = \min(X_1, X_2)$?

$$g_1(y) = n[1-F(y)]^{n-1} f(y) = 2\left[1-(1-e^{-y/10})\right]^1\left(\tfrac{1}{10}e^{-y/10}\right) = 2\left[e^{-y/10}\right]\left(\tfrac{1}{10}e^{-y/10}\right) = \underbrace{\tfrac{1}{5}e^{-y/5}}_{Y_1 \sim \text{Exp}(\lambda = \frac{1}{5})} \text{ for } y>0$$

(b) What is the expected value of Y_1?

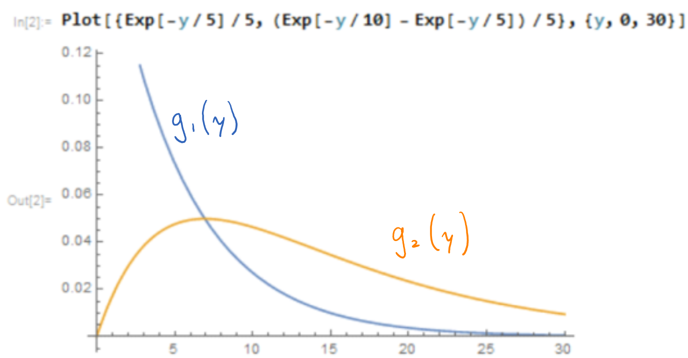
$$E(Y_1) = 5$$

(c) What is the pdf of $Y_2 = \max(X_1, X_2)$? What is $E(Y_2)$?

$$g_2(y) = n\left[F(y)\right]^{n-1} f(y) = 2\left[1-e^{-y/10}\right]^1\left(\tfrac{1}{10}e^{-y/10}\right) = \tfrac{2}{10}\left(e^{-y/10} - e^{-y/5}\right) = \tfrac{1}{5}\left(e^{-y/10} - e^{-y/5}\right) \text{ for } y>0$$

$$E(Y_2) = \int_0^\infty y \cdot \tfrac{1}{5}\left(e^{-y/10} - e^{-y/5}\right) dy = 15$$

pdfs of Y_1 and Y_2:



2. Let X_1, X_2, X_3 be iid Exp$\left(\frac{1}{10}\right)$. What is the expected value of the sample median?

Sample median is Y_2 $(n=3,\ i=2)$

$$g_2(y) = \frac{3!}{1!\,1!}\left[1-e^{-y/10}\right]^1\left[e^{-y/10}\right]^1\left(\tfrac{1}{10}e^{-y/10}\right) = \tfrac{6}{10}\left[1-e^{-y/10}\right]e^{-y/5} = \tfrac{3}{5}\left(e^{-y/5} - e^{-3y/10}\right) \text{ for } y>0$$

$$E(Y_2) = \int_0^\infty y \cdot \tfrac{3}{5}\left(e^{-y/5} - e^{-3y/10}\right) dy = \frac{25}{3}$$

3. Let X_1, X_2, X_3 be iid Unif[0,1]. What is the probability that the sample median is between $\frac{1}{4}$ and $\frac{3}{4}$?

$n=3$, Unif$[0,1]$ has pdf $f(x) = 1$, cdf $F(x) = x$, for $0 \le x \le 1$

density of the sample median:

$$g_2(y) = \frac{3!}{(2-1)!(3-2)!}\, y(1-y) = 6y(1-y) \quad \text{for } 0 \le y \le 1$$

Thus, $P\left(\frac{1}{4} < Y_2 < \frac{3}{4}\right) = \int_{\frac{1}{4}}^{\frac{3}{4}} (6y - 6y^2)\,dy = \left[3y^2 - 2y^3\right]_{\frac{1}{4}}^{\frac{3}{4}} = \frac{11}{16}$

4. Let n be a positive odd integer and let $X_1, X_2, \dots, X_n$ be iid Unif[0,1]. What is the smallest n such that the sample median is between 0.4 and 0.6 with probability greater than $\frac{1}{2}$?

median is Y_i with $i = \frac{n+1}{2}$

density: $g_i(y) = \frac{n!}{(i-1)!(n-i)!}\, y^{i-1}(1-y)^{n-i} = \frac{n!}{\left(\frac{n-1}{2}\right)!^2}\, y^{\frac{n-1}{2}}(1-y)^{\frac{n-1}{2}}$

Use Mathematica to compute $\int_{0.4}^{0.6} g_i(y)\,dy$ for various n.

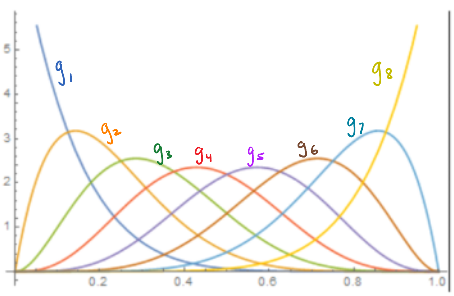
The smallest odd n such that $\int_{0.4}^{0.6} g_i(y)\,dy > \frac{1}{2}$ is $n = 11$.

5. Let $X_1, \dots, X_8$ be iid Unif[0,1].

(a) Make a plot of the pdfs of all eight order statistics.

```
In[16]:= n = 8;
         pdfs = Table[n! / ((i - 1)! (n - i)!) y^(i - 1) (1 - y)^(n - i), {i, 1, n}]
Out[17]= {8 (1 - y)^7, 56 (1 - y)^6 y, 168 (1 - y)^5 y^2, 280 (1 - y)^4 y^3, 280 (1 - y)^3 y^4, 168 (1 - y)^2 y^5, 56 (1 - y) y^6, 8 y^7}

In[18]:= Plot[pdfs, {y, 0, 1}]
Out[18]=
```



(b) What are the expected values of all eight order statistics?

$$E(Y_i) = \frac{i}{9}$$