Bivariate Transformations

Random variables X_1 and X_2 have joint density $f(x_1, x_2)$ and $Y = g(X_1, X_2)$. What is the density of Y?

cdf method:

(a) Identify the possible values of Y.

(b) For a fixed value y, sketch $Y = y$ in the x_1x_2-plane.

(c) Find the region R in the x_1x_2-plane where $Y \le y$.

(d) Find the cdf $F_Y(y)$ by integrating $f(x_1, x_2)$ over R.

(e) Differentiate F_Y to obtain the density $f_Y(y)$.

Bivariate Transformation Theorem

Let X_1 and X_2 have joint density $f(x_1, x_2)$.

Let $Y_1 = u_1(X_1, X_2)$ and $Y_2 = u_2(X_1, X_2)$,

with inverse transformation $X_1 = v_1(Y_1, Y_2)$ and $X_2 = v_2(Y_1, Y_2)$.

Let M be the Jacobian matrix:

$$M = \begin{bmatrix} \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial y_2} \\ \frac{\partial v_2}{\partial y_1} & \frac{\partial v_2}{\partial y_2} \end{bmatrix}$$

Then the joint density of Y_1 and Y_2 is given by

$$g(y_1, y_2) = f(v_1(y_1, y_2), v_2(y_1, y_2)) \cdot |\det(M)|.$$

Ch. 3: $g_Y(y) = f_X(h(y)) \cdot |h'(y)|$

EXAMPLE: POLAR COORDINATES

Let X and Y have joint density $f(x,y)$.
Let (R, Θ) be the polar coordinates of (X, Y).

$$X = R\cos(\Theta)$$
$$Y = R\sin(\Theta)$$

Then the joint density of R and Θ is

$$g(r, \theta) = f(r\cos\theta, r\sin\theta) \cdot r$$

ORDER STATISTICS

Let $X_1, X_2, \dots, X_n$ be iid continuous rvs with pdf $f(x)$ and cdf $F(x)$. The order statistics of this random sample are $Y_1, Y_2, \dots, Y_n$, where

Y_i = the i^{th} smallest value among $X_1, X_2, \ldots, X_n$.

MIN: $Y_1 = \min(X_1, \ldots, X_n)$ has pdf $g_1(y) = n\left[1 - F(y)\right]^{n-1} \cdot f(y)$

MAX: $Y_n = \max(X_1, \ldots, X_n)$ has pdf $g_n(y) = n\left[F(y)\right]^{n-1} \cdot f(y)$

i^{th} Smallest: Y_i has pdf $g_i(y) = \dfrac{n!}{(i-1)!\,(n-i)!}\left[F(y)\right]^{i-1}\left[1 - F(y)\right]^{n-i} \cdot f(y)$