Solutions to limit theorem problems from the previous class

6. Suppose you flip a fair coin *lots* of times. What does the Law of Large Numbers say about the numbers of heads and tails you will observe?

These numbers must approach 50% of the number of flips.

7. Suppose that a certain casino game costs \$1 to play, and the expected winnings per game are \$0.98. What does the Law of Large Numbers say about your winnings if you play the game lots of times?

If you play n games (where n is large), you will lose about \$0.02 $\cdot$ n, and this gets more accurate the larger n gets.

8. Fifty real numbers are each rounded to the nearest integer and then summed. If the indivdual round-off errors are uniformly distributed over $(-0.5, 0.5)$, then approximate the probability that the resultant sum differs from the exact sum by no more than 3.

Let the individual errors be $X_1, X_2, \ldots, X_{50} \sim \text{Unif}\left(\frac{-1}{2}, \frac{1}{2}\right)$, which have mean 0 and standard deviation $\frac{1}{\sqrt{12}}$.

Then the sum $T_{50} = \sum_{i=1}^{50} X_i$ is approximately normal with mean 0 and standard deviation $\frac{\sqrt{50}}{\sqrt{12}}$.

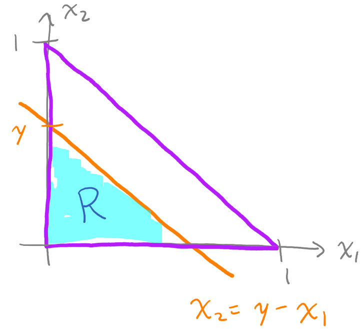
So $P(|T_{50}| > 3) = 2 \cdot \underbrace{P(T_{50} < -3)}_{\text{compute using normal cdf}} \approx 0.1416.$

9. Suppose that a fair coin is tossed 1000 times. If the first 100 tosses result in heads, what proportion of heads would you expect on the remaining 900 tosses? Interpret the statement "The law of large numbers swamps, but it does not compensate."

We expect about 450 heads in the remaining 900 tosses.

Any given observations do not change the probabilities for later tosses. However, even a very unusual sequence of heads will be insignificant in the long run as the proportion of heads will converge to $\frac{1}{2}$.

1. Let X_1 and X_2 be uniformly distributed over the region of the x_1x_2-plane defined by $0 \le x_1$, $0 \le x_2$, and $x_1 + x_2 \le 1$. Let $Y = X_1 + X_2$. Use the following steps to find the density of Y.



(a) Identify the possible values of Y.

$0 \le Y \le 1$

(b) Sketch the graph $Y = y$ in the x_1x_2-plane.

For $y \in [0,1]$: $Y = X_1 + X_2 = y \Rightarrow x_2 = y - x_1$

(c) Find the region R in the x_1x_2-plane where $Y \le y$.

$Y \le y \Rightarrow x_2 \le y - x_1$, which is the region shaded blue

(d) Find the cdf $F_Y(y)$ by integrating the joint density of X_1 and X_2 over the region R.

$$F_Y(y) = P(X_2 \le y - X_1) = \iint_R f(x_1, x_2)\, dA$$

$$= \iint_R 2\, dA = 2 \cdot \text{Area}(R) = y^2$$

(e) Differentiate $F_Y(y)$ to obtain the density $f_Y(y)$.

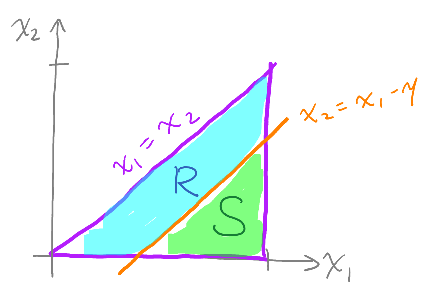
$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}(y^2) = 2y \quad \text{for } 0 \le y \le 1.$$

2. Let X_1 and X_2 have joint density $f(x_1, x_2) = 3x_1$, for $0 \le x_2 \le x_1 \le 1$. Let $Y = X_1 - X_2$. Use the following steps to find the density of Y.

First, note that $0 < Y < 1$

Fix $y \in [0,1]$. Then $y = x_1 - x_2$, or

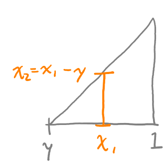
$\underline{x_2 = x_1 - y.}$



So $Y = X_1 - X_2 \le y$ is the region R.

Now integrate to find the cdf of Y:

$$F_Y(y) = P(Y \le y) = P(X_1 - X_2 \le y) = \iint_R f(x_1, x_2)\, dA = 1 - \iint_S f(x_1, x_2)\, dA$$

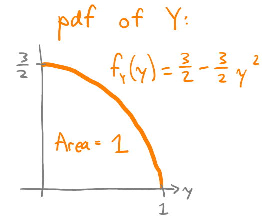


$$= 1 - \int_y^1 \int_0^{x_1 - y} 3x_1\, dx_2\, dx_1 = 1 - \int_y^1 3x_1(x_1 - y)\, dx_1$$

$$= 1 - \left[x_1^3 - \tfrac{3}{2}x_1^2 y\right]_{x_1=y}^{x_1=1} = 1 - \left[\left(1-\tfrac{3}{2}y\right) - \left(y^3 - \tfrac{3}{2}y^3\right)\right] = \tfrac{3}{2}y - \tfrac{1}{2}y^3$$

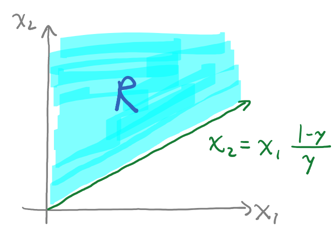
Differentiate to obtain the pdf of Y:

$$f_Y(y) = \frac{d}{dy}\left[\tfrac{3}{2}y - \tfrac{1}{2}y^3\right] = \frac{3}{2} - \frac{3}{2}y^2 \quad \text{for } 0 \le y \le 1$$



3. The joint density of X_1 and X_2 is $f(x_1, x_2) = 4e^{-2(x_1+x_2)}$ for $X_1 > 0$ and $X_2 > 0$. Find the density of $Y = \frac{X_1}{X_1+X_2}$.

First, note that $0 \le Y \le 1$.



For $y \in [0, 1]$: $\quad Y = y \Rightarrow \frac{X_1}{X_1+X_2} = y \Rightarrow X_1 = X_1 y + X_2 y$

$$\Rightarrow \frac{X_1(1-y)}{y} = X_2$$

$$Y \le y \Rightarrow x_1 \frac{1-y}{y} \le x_2$$

Then: $\quad F_Y(y) = \iint_R f(x_1,x_2)\,dx_2\,dx_1 = \int_0^\infty \int_{\frac{1-y}{y}x_1}^\infty 4e^{-2x_1-2x_2}\,dx_2\,dx_1 = \int_0^\infty 2e^{-2x_1/y}\,dx_1$

inner integral:

$$\int_{\frac{1-y}{y}x_1}^\infty e^{-2x_2}\,dx_2 = \frac{-1}{2}e^{-2x_2}\Big|_{\frac{1-y}{y}x_1}^\infty = \frac{1}{2}e^{-2x_1\frac{1-y}{y}}$$

$$F_Y(y) = -y e^{-2x_1/y}\Big|_0^\infty = y \qquad \text{so } F_Y(y) = y \quad \text{for } 0 \le y \le 1.$$

Thus, $f_Y(y) = 1$ for $0 \le y \le 1$.

NOTE: X_1, X_2 are iid Exp(2).
Y is the proportion of the sum $X_1 + X_2$ due to X_1.

We will come back to these problems next time:

4. Suppose that X_1 and X_2 are iid Unif[0,1]. Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.

(a) Find the region of possible values of the pair (Y_1, Y_2).

(b) Find the inverse transformation functions v_1 and v_2 such that $X_1 = v_1(Y_1, Y_2)$ and $X_2 = v_2(Y_1, Y_2)$.

(c) Use the transformation theorem to find the joint pdf of Y_1 and Y_2.

5. Let (X, Y) be a random point in the plane, where X and Y are independent standard normal random variables. Let (R, Θ) be the polar coordinates of (X, Y). Find the joint density of R and Θ. Then find the marginal densities of R and Θ. What is the probability that the point (X, Y) lies in a circle of radius 1 centered at the origin?