From last time:

4. Let X and Y have joint pdf $f(x, y) = 3x + 3y$ for $0 \le x$, $0 \le y$, and $x + y \le 1$.

(a) Sketch the joint pdf and verify that the volume underneath is 1.

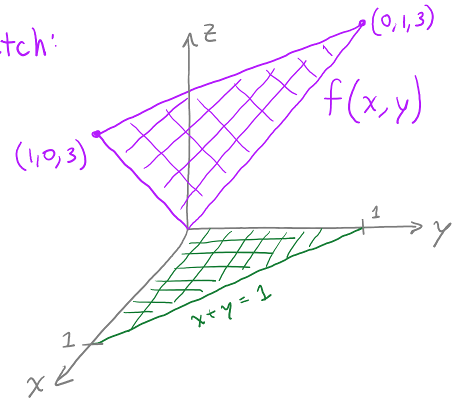
Volume:

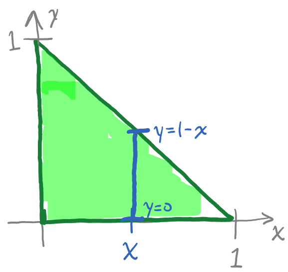
$$\int_0^1 \int_0^{1-x} (3x+3y)\, dy\, dx = 1$$

note the bounds of integration

$0 \le x \le 1, \quad 0 \le y \le 1-x$

sketch:





To compute the integral using Mathematica:

```
Integrate[3 x + 3 y, {x, 0, 1}, {y, 0, 1 - x}]
```

(b) What values of X and Y are most likely? What values are not so likely?

X and Y are likely to have a sum close to 1.

X and Y are not likely to both be near zero.

(c) Compute the following, using technology to evaluate integrals:

- $E(X + Y)$

$$E(X+Y) = \int_0^1 \int_0^{1-x} (x+y)(3x+3y)\, dy\, dx = \boxed{\frac{3}{4}}$$

```
Integrate[(x + y) (3 x + 3 y), {x, 0, 1}, {y, 0, 1 - x}]
```

- $E(XY)$

$$E(XY) = \int_0^1 \int_0^{1-x} (xy)(3x+3y)\, dy\, dx = \boxed{\frac{1}{10}}$$

- $E(X)$

$$f_X(x) = \int_0^{1-x} (3x+3y)\, dy = \frac{3}{2}(1-x^2), \quad 0 \le x \le 1$$

$$E(X) = \int_0^1 x \cdot \frac{3}{2}(1-x^2)\, dx = \boxed{\frac{3}{8}}$$

- $E(Y)$

$$f_Y(y) = \int_0^{1-y} (3x+3y)\,dx = \frac{3}{2}(1-y^2), \quad 0 \le y \le 1,$$

$$E(Y) = \int_0^1 y \cdot \frac{3}{2}(1-y^2)\,dy = \boxed{\frac{3}{8}}$$

(d) What is $\mathrm{Cov}(X,Y)$?

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{10} - \frac{3}{8}\cdot\frac{3}{8} = \boxed{\frac{-13}{320}}$$

New worksheet for today:

1. How do $E(X)$ and $E(Y)$ relate to $E(X+Y)$ and $E(XY)$? Does independence play a role?

$E(X+Y) = E(X) + E(Y)$ by linearity of expectation,
regardless of whether X and Y are independent or not.

If X and Y are independent, then $E(XY) = E(X)E(Y)$.
(The converse is not true!)

2. Let $X \sim \mathrm{Unif}[-1,1]$ and $Y = X^2$.

(a) Compute $E(X)$, $E(Y)$, and $E(XY)$. Does $E(XY) = E(X)E(Y)$?

$E(X) = 0$, $\quad E(Y) = E(X^2) = \int_{-1}^{1} x^2 \cdot \frac{1}{2}\,dx = \frac{1}{6}x^3\Big|_{-1}^{1} = \frac{1}{3}$

$E(XY) = E(X^3) = \int_{-1}^{1} x^3 \cdot \frac{1}{2}\,dx = \frac{1}{8}x^4\Big|_{-1}^{1} = 0$

Yes, $E(XY) = E(X)E(Y)$

(b) Are X and Y independent? Why or why not?

No: the value of X determines Y.

ANOTHER EXAMPLE: $U \sim \mathrm{Unif}[0, 2\pi]$, $X = \cos(U)$, $Y = \sin(U)$

I. Two standard, fair dice are rolled. Let X_1 and X_2 be the numbers that appear on the dice.

II. An urn contains balls labeled 1, 2, 3, 4, 5, 6. Let Y_1 and Y_2 be the numbers on two balls drawn without replacement from the urn.

3. What is the distribution of X_i? How about the distribution of Y_i?

X_i and Y_i are both uniformly distributed on $\{1, 2, 3, 4, 5, 6\}$.

4. What are $E(X_i)$ and $\text{Var}(X_i)$? How about $E(Y_i)$ and $\text{Var}(Y_i)$?

$E(X_i) = \frac{7}{2}$, $\quad E(X_i)^2 = \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6}$

$$\text{Var}(X_i) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

Since X_i and Y_i have the same distribution,

$E(Y_i) = \frac{7}{2}$ and $\text{Var}(Y_i) = \frac{35}{12}$

5. What are $E(X_1 + X_2)$ and $\text{Var}(X_1 + X_2)$?

By linearity, $E(X_1 + X_2) = E(X_1) + E(X_2) = 7$

Since X_1 and X_2 are independent,

$$\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2) = \frac{35}{6}$$

6. What and $E(Y_1 + Y_2)$ and $\text{Var}(Y_1 + Y_2)$?

By linearity, $E(Y_1 + Y_2) = E(Y_1) + E(Y_2) = 7$

Since Y_1 and Y_2 are dependent:

$$\text{Var}(Y_1 + Y_2) = \text{Var}(Y_1) + \text{Var}(Y_2) + 2\,\text{Cov}(Y_1, Y_2)$$
$$= \frac{35}{12} + \frac{35}{12} + 2\left(\frac{-7}{12}\right) = \frac{56}{12} = \frac{14}{3}$$

$$\text{Cov}(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2) = \frac{35}{3} - \frac{7}{2}\cdot\frac{7}{2} = -\frac{7}{12}$$

Possible products:

	1	2	3	4	5	6
1	□	2	3	4	5	6
2	2	□	6	8	10	12
3	3	6	□	12	15	18
4	4	8	12	□	20	24
5	5	10	15	20	□	30
6	6	12	18	24	30	□

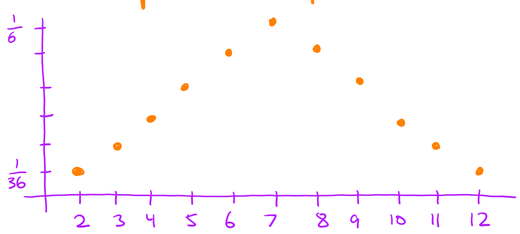
All equally likely.

$$E(Y_1Y_2) = \frac{1}{15}(2+3+4+5+6+6+8+10+12+12+15+18+20+24+30)$$

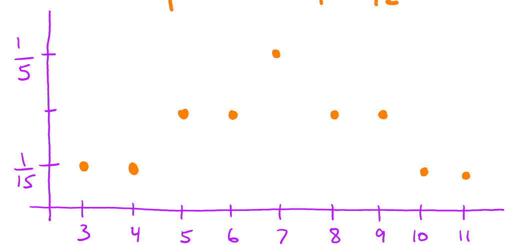
$$= \frac{175}{15} = \frac{35}{3}$$

7. Sketch the pmfs of $X_1 + X_2$ and $Y_1 + Y_2$. How does this help make sense of the means and variances that you found for these sums?

pmf of $X_1 + X_2$



pmf of $Y_1 + Y_2$



The means are the same, but the distribution of $X_1 + X_2$ is more spread out and thus has larger variance.

8. Generalize to rolls of n dice: find $E(X_1 + \cdots + X_n)$ and $\mathrm{Var}(X_1 + \cdots + X_n)$.

$$E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n) = \frac{7n}{2}$$

by independence, $\mathrm{Var}(X_1 + \cdots + X_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n) = \frac{35n}{12}$

9. Similarly, generalize to choosing n balls from the urn. Find $E(Y_1 + \cdots + Y_n)$ and $\mathrm{Var}(Y_1 + \cdots + Y_n)$.

Now $n \leq 6$.

As before, $E(Y_1 + \cdots + Y_n) = \frac{7n}{2}$

However, now $\mathrm{Var}(Y_1 + \cdots + Y_n) = \sum_{i=1}^{n} \mathrm{Var}(Y_i) + 2\sum_{i<j} \mathrm{Cov}(Y_i, Y_j)$

$$= \frac{35n}{12} + 2\,\frac{n^2-n}{2}\left(-\frac{7}{12}\right) = \frac{35n - 7n^2 + 7n}{12} = \frac{42n - 7n^2}{12} = \frac{7n(6-n)}{12}$$

Note that if $n = 6$, the variance is zero.