Worksheet Solutions
Math 262 • 15 November 2022

1. A cafeteria has three meal options: pizza, burgers, and salad bar. Three students each choose one option independently at random (equally likely to choose any option). Let X be the number (of the 3) who choose pizza, and let Y be the number who choose the salad bar.

(a) What is the joint pmf of X and Y? What are the marginal pmfs of X and Y?

e.g.:

$$p(0,0) = P\left(\begin{smallmatrix}\text{student 1}\\ \text{chooses burger}\end{smallmatrix}\right)\cdot P\left(\begin{smallmatrix}\text{student 2}\\ \text{chooses burger}\end{smallmatrix}\right)\cdot P\left(\begin{smallmatrix}\text{student 3}\\ \text{chooses burger}\end{smallmatrix}\right)$$

$$= \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} = \frac{1}{27}$$

joint pmf:

x

marginal pmf:

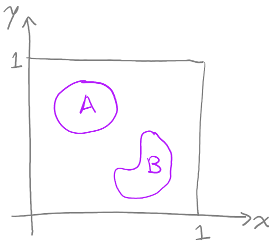
Y	0	1	2	3	$p_Y(y)$
0	$\frac{1}{27}$	$\frac{3}{27}$	$\frac{3}{27}$	$\frac{1}{27}$	$\frac{8}{27}$
1	$\frac{3}{27}$	$\frac{6}{27}$	$\frac{3}{27}$	0	$\frac{12}{27}$
2	$\frac{3}{27}$	$\frac{3}{27}$	0	0	$\frac{6}{27}$
3	$\frac{1}{27}$	0	0	0	$\frac{1}{27}$
marginal pmf: $p_X(x)$	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$	

(b) Are X and Y independent? Why or why not?

No, since knowledge of one affects the probabilities of the other.

2. Suppose a particle is randomly located in the square $0 \le x \le 1$, $0 \le y \le 1$. That is, if two regions within the square have equal area, then the particle is equally likely to be in either region. Let (X, Y) be the coordinates of the particle.

(a) What is the joint density function of X and Y?



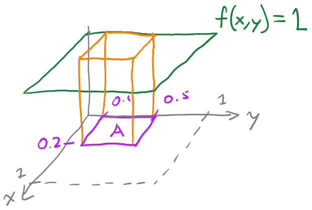
If Area(A) = Area(B), then the particle is equally likely to be in A or B.

Thus, $P((X,Y)\in A) = k\cdot \text{Area}(A) = \iint_A f(x,y)\,dA$

The joint density is then $f(x,y) = k$.

Since $\int_0^1\int_0^1 k\,dx\,dy = k = 1$, we have $\underbrace{f(x,y) = 1 \text{ for } 0\le x\le 1 \text{ and } 0 \le y \le 1.}_{\text{2-D Uniform Distribution}}$

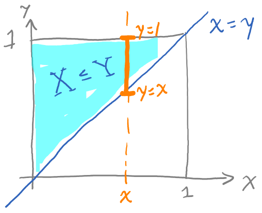
(b) Find $P(X \le 0.2, 0.1 \le Y \le 0.5)$.



$$P(X \le 0.2,\ 0.1 \le Y \le 0.5) = \int_{0.1}^{0.5}\int_0^{0.2} f(x,y)\,dx\,dy$$

$$= (0.2)(0.4)(1) = 0.08$$

(c) Find $P(X \le Y)$.



$$P(X \le Y) = \iint_A 1 \; dA = \frac{1}{2}$$

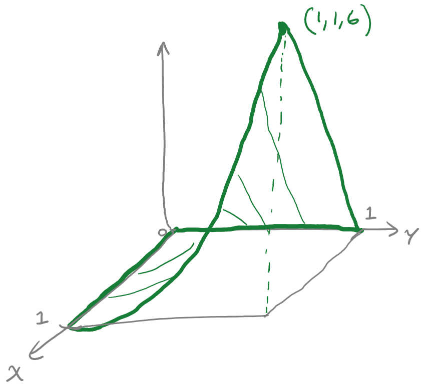
$$= \int_0^1 \int_x^1 1 \; dy\, dx$$

(d) Are X and Y independent? Why or why not?

Yes: $f(x,y) = f_X(x)\, f_Y(y)$ for $0 \le x \le 1$, $0 \le y \le 1$.

3. Let X and Y have joint pdf $f(x,y) = 6xy^2$ for $0 \le x \le 1$ and $0 \le y \le 1$.

(a) Verify that $f(x,y)$ is a joint pdf.



$f(x,y) \ge 0$ for $0 \le x \le 1$, $0 \le y \le 1$ and

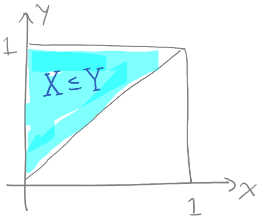
$$\int_0^1\int_0^1 6xy^2\, dx\, dy = 6 \int_0^1 x\, dx \int_0^1 y^2 dy$$

$$= 6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = 1$$

(b) What is $P(X \le Y)$?

$$P(X \le Y) = \int_0^1 \int_x^1 6xy^2\, dy\, dx = \int_0^1 (2x - 2x^4)\, dx = 1 - \frac{2}{5} = \frac{3}{5}$$

$$\int_x^1 6xy^2\, dy = 2xy^3 \Big|_{y=x}^{y=1} = 2x - 2x^4$$



(c) Find $f_X(x)$ and $f_Y(y)$.

$$f_X(x) = \int_0^1 6xy^2\, dy = 2xy^3 \Big|_{y=0}^{y=1} = 2x \quad \text{for } 0 \le x \le 1$$

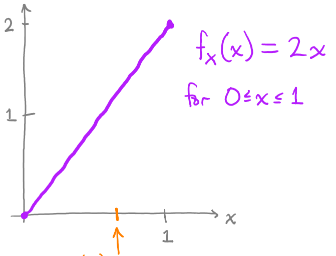
$$f_Y(y) = \int_0^1 6xy^2\, dx = 3x^2y^2 \Big|_{x=0}^{x=1} = 3y^2 \quad \text{for } 0 \le y \le 1$$

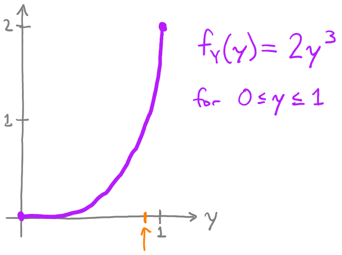
(d) Are X and Y independent? Why or why not?

Yes: $f(x,y) = f_X(x)\, f_Y(y)$

$6xy^2 = (2x)(3y^2)$ for $0 \le x \le 1$, $0 \le y \le 1$

(e) Sketch the marginal pdfs $f_X(x)$ and $f_Y(y)$. What would you estimate to be the means $E(X)$ and $E(Y)$?





E(X) is about here
maybe $E(X) \approx 0.7$?

E(Y) should be close to 1
maybe $E(Y) \approx \frac{7}{8}$?

(f) Compute $E(X)$ and $E(Y)$.

$$f_X(x) = 2x, \quad 0 \le x \le 1, \quad \text{so} \quad E(X) = \int_0^1 x \cdot 2x \, dx = \frac{2}{3} x^3 \Big|_0^1 = \boxed{\frac{2}{3}}$$

$$f_Y(y) = 3y^2, \quad 0 \le y \le 1, \quad \text{so} \quad E(Y) = \int_0^1 y \cdot 3y^2 \, dy = \frac{3}{4} y^4 \Big|_0^1 = \boxed{\frac{3}{4}}$$

(g) Compute $E(X+Y)$ in two different ways.

I. $E(X+Y) = E(X) + E(Y)$ ← linearity of expected value

$$E(X+Y) = E(X) + E(Y) = \frac{2}{3} + \frac{3}{4} = \frac{17}{12}$$

II. $E(X+Y) = \int_0^1\int_0^1 (x+y)\, 6xy^2 \, dy\, dx$ ← expected value of a function of X and Y

$$E(X+Y) = \int_0^1 \int_0^1 (x+y)\, 6xy^2 \, dy\, dx = \boxed{\frac{17}{12}}$$

Mathematica: $\texttt{Integrate[(x + y) 6 x y\^{}2, }\underbrace{\texttt{\{x, 0, 1\}}}_{\text{outer bounds}}\texttt{, }\underbrace{\texttt{\{y, 0, 1\}}}_{\text{inner bounds}}\texttt{]}$

(h) Now compute $E(XY)$.

$$E(XY) = \int_0^1 \int_0^1 (xy)\, 6xy^2 \, dy\, dx = \boxed{\frac{1}{2}}$$

Note that for this problem, $E(XY) = E(X)E(Y)$.

(i) What are the values of $\text{Cov}(X,Y)$ and $\text{Corr}(X,Y)$? (Try to do this without evaluating any more integrals.)

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{2} - \frac{2}{3} \cdot \frac{3}{4} = 0$$

$$\text{Corr}(X,Y) = 0$$

4. Let X and Y have joint pdf $f(x,y) = 3x + 3y$ for $0 \leq x$, $0 \leq y$, and $x + y \leq 1$.

(a) Sketch the joint pdf and verify that the volume underneath is 1.

We will do this on Thursday.

(b) What values of X and Y are most likely? What values are not so likely?

(c) Compute the following, using technology to evaluate integrals:

- $E(X+Y)$
- $E(XY)$
- $E(X)$
- $E(Y)$

(d) What is $\text{Cov}(X,Y)$?