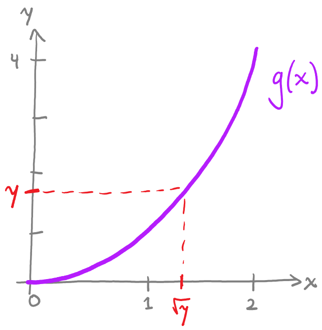
1. Let X have density $f_X(x) = \frac{x}{2}$ for $0 \le x \le 2$, and let $Y = X^2$. What is the density of Y?

Graph of Transformation



NOTE: Y takes values $0 \le y \le 4$

Find cdf of Y: for $y \in [0,4]$:

$$F_Y(y) = P(Y \le y) = P(X \le \sqrt{y})$$

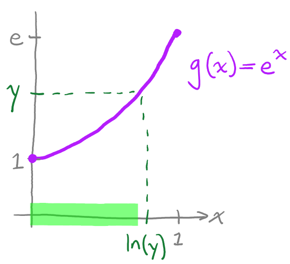
$$= \int_0^{\sqrt{y}} \frac{x}{2}\,dx = \frac{x^2}{4}\Big|_0^{\sqrt{y}} = \frac{y}{4} - 0 = \frac{y}{4}$$

Differentiate to obtain the pdf of Y:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(\frac{y}{4}\right) = \frac{1}{4} \quad \text{for } 0 \le y \le 4$$

2. Let X have density $f_X(x) = 2x$ for $0 \le x \le 1$, and let $Y = e^X$. What is the density of Y?

(a) Sketch the transformation $y = e^x$ and identify the possible values of Y.



Possible values of Y:

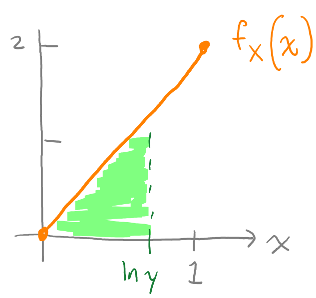
$$1 \le Y \le e$$

(b) Find the cdf of Y, and differentiate to obtain the pdf.

find the cdf of Y: for $y \in [1, e]$,

$$F_Y(y) = P(Y \le y) = P(e^X \le y) = P(X \le \ln(y))$$

$$= \int_0^{\ln y} 2x\,dx = x^2\Big|_0^{\ln y} = (\ln y)^2$$



differentiate to obtain the pdf of Y:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}(\ln y)^2 = 2\ln(y) \cdot \frac{1}{y}$$

$$f_Y(y) = \frac{2}{y}\ln(y) \quad \text{for } \underline{1 \le y \le e}$$

bounds are important!

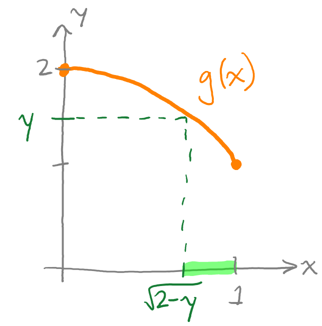
(c) Confirm that you obtain the same answer via the Transformation Theorem.

$g(x) = e^x$, which is strictly increasing on $0 \le x \le 1$

inverse is $h(y) = \ln y$, which is differentiable

thus: $f_Y(y) = f_X(h(y))\,|h'(y)| = 2(\ln y)\left|\frac{1}{y}\right| = \frac{2}{y}\ln y$ for $1 \le y \le e$

3. Let X have density $f_X(x) = 2x$ for $0 \le x \le 1$, and let $Y = 2 - X^2$. What is the density of Y?



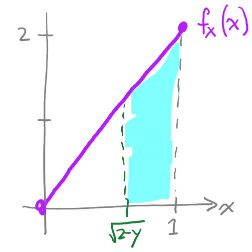
$$\left[\begin{array}{c} 0 \le Y \le y \\ \text{iff} \\ \sqrt{2-y} \le X \le 1 \end{array}\right]$$

Note that $1 \le Y \le 2$. Then for $y \in [1, 2]$:

$$F_Y(y) = P(Y \le y) = P(2 - X^2 \le y) = P(\sqrt{2-y} \le X)$$

$$= \int_{\sqrt{2-y}}^{1} f_X(x)\,dx = \int_{\sqrt{2-y}}^{1} 2x\,dx$$

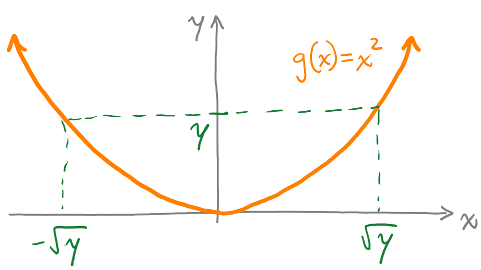
$$= x^2 \Big|_{\sqrt{2-y}}^{1} = 1 - (2-y) = y - 1$$



Then: $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}(y-1) = 1$

So: $f_Y(y) = 1$ for $1 \le y \le 2$

4. Let $X \sim N(0,1)$ and $Y = X^2$. What is the distribution of Y?



Note that $Y \ge 0$.

For $y \ge 0$, $F_Y(y) = P(Y \le y) = P(X^2 \le y)$

$$= P(\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx.$$

Then $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{\sqrt{2\pi}}\left(e^{-y/2}\frac{1}{2\sqrt{y}} + e^{-y/2}\frac{1}{2\sqrt{y}}\right) = \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}}$

by the Fundamental Theorem of Calculus

Thus $f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-y/2}$ for $y \ge 0$.

This is the pdf of the Gamma$(\alpha = \frac{1}{2}, \beta = 2)$ distribution,
which is also the chi-square distribution with 1 degree of freedom.

5. Let $U \sim$ Unif[0,1] and X have pdf $f(x)$.
We wish to find a transformation from Unif[0,1] to the distribution of X. In other words, we want to find a function g such that if $X = g(U)$, then the pdf of X is $f(x)$.

(a) If we want to apply the Transformation Theorem, what do we have to assume about g?

We must assume that g is invertible, and that g^{-1} is differentiable.

(b) Apply the Transformation Theorem to the situation described above. How does the theorem allow you to find a transformation function g?

We want: $f_X(x) = f_U(h(x)) \cdot |h'(x)|$ where $h = g^{-1}$.

Since $U \sim$ Unif[0,1], the pdf of U is

$f_U(u) = 1$ for $0 \le u \le 1$ and $f_U(u) = 0$ otherwise.

Assuming $0 \le h(x) \le 1$, we want $f_X(x) = 1 \cdot |h'(x)|$.

This would work if $h'(x) = f_X(x)$.

Integrate to obtain $h(x) = \int_0^x f_X(t)\,dt = F_X(x)$, the cdf of X.

Thus, $h = g^{-1}$, we find that $g(u) = F_X^{-1}(u)$.

(c) Does your function g satisfy the assumptions of the Transformation Theorem? Explain.

Suppose $f_X(x)$ is positive on some interval I and $f_X(x) = 0$ otherwise.

Then $F_X(x)$ is strictly increasing on I, and thus invertible on I

(F_X is one-to-one on I) so F_X^{-1} exists.

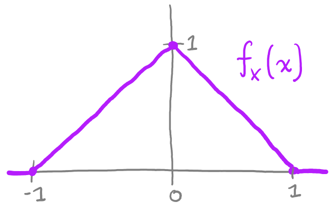
So $g = F_X^{-1}$ and $h = F_X$.

Since $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$, F_X is differentiable by the Fundamental Theorem of Calculus.

Thus, the assumptions of the Transformation Theorem are satisfied.

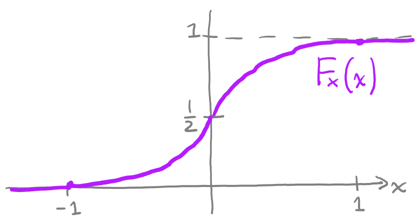
6. Let X have density given by $f_X(x) = \begin{cases} x+1 & \text{if } -1 \le x \le 0, \\ 1-x & \text{if } 0 < x \le 1, \\ 0 & \text{otherwise.} \end{cases}$

(a) Sketch the pdf of X.



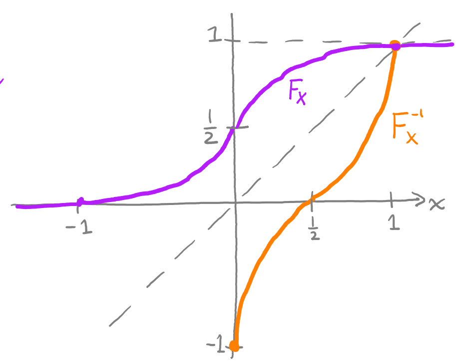
(b) Find a formula for the cdf $F_X(x)$. Also sketch $F_X(x)$.

$$F_X(x) = \int_{-1}^{x} f_X(t)\,dt = \begin{cases} \int_{-1}^{x} (t+1)\,dt = \left[\frac{1}{2}t^2 + t\right]_{-1}^{x} = \frac{x^2}{2} + x + \frac{1}{2} & \text{if } -1 \le x \le 0 \\ \frac{1}{2} + \int_{0}^{x} (1-t)\,dt = \frac{1}{2} + \left[t - \frac{1}{2}t^2\right]_{0}^{x} = \frac{1}{2} + x - \frac{x^2}{2} & \text{if } 0 \le x \le 1 \end{cases}$$



(c) Sketch the inverse of $F_X^{-1}(x)$. Then find a formula for $F_X^{-1}(x)$.

To sketch the inverse, flip F_X across the line $y = x$.



To find a formula for F_X^{-1}, we must consider each piece of F_X separately. Let $u = F_X(x)$ for $-1 \le x \le 1$.

If $-1 \le x \le 0$, then $0 \le u \le \frac{1}{2}$.

In this case: $u = \frac{x^2}{2} + x + \frac{1}{2}$

$2u = x^2 + 2x + 1$

$0 = x^2 + 2x + 1 - 2u$

so: $x = \frac{-2 \pm \sqrt{4 - 4(1-2u)}}{2} = -1 + \sqrt{2u}$

If $0 < x \le 1$, then $\frac{1}{2} < u \le 1$.

In this case: $u = \frac{1}{2} + x - \frac{x^2}{2}$

$-2u = -1 - 2x + x^2$

$0 = x^2 - 2x - 1 + 2u$

so: $x = \frac{2 \pm \sqrt{4 - 4(2u-1)}}{2} = 1 - \sqrt{2 - 2u}$

Thus, the inverse of $u = F_X(x)$ is:

$$F_X^{-1}(u) = \begin{cases} -1 + \sqrt{2u} & \text{if} \quad 0 \le u \le \frac{1}{2} \\ 1 - \sqrt{2-2u} & \text{if} \quad \frac{1}{2} < u \le 1 \end{cases}$$

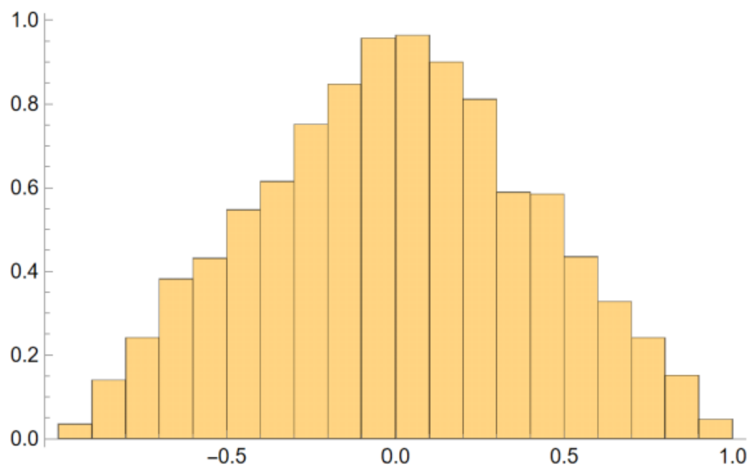
(d) Write a program to simulate values of X. Simulate thousands of values and make a histogram. Does your histogram look like the density you sketched in part (a)?

Mathematica:

```
simX[] := Module[{},
  u = RandomReal[];
  If[u ≤ 1 / 2, x = -1 + Sqrt[2 u], x = 1 - Sqrt[2 - 2 u]];
  Return[x]
 ]
```

```
xvals = Table[simX[], 10000]
```

```
Histogram[xvals, 20, "PDF"]
```



R:

```
simX <- function(){
  u = runif(1)
  if(u <= 0.5){
    return(-1 + sqrt(2*u))
  } #else
  return(1 - sqrt(2 - 2*u))
}

xvals = replicate(10000, simX())
hist(xvals, freq=FALSE)
```

Yes, this looks like part (a).