1. Suppose that 45% of the phone calls you receive are scam calls. Assume that the probability of a scam call is independent from one call to the next. Let X be the number of calls you receive until (and including) the next scam call.
   Express each of the following probabilities in terms of X. Then compute each probability.

(a) The probability that *none* of the first 4 calls are scam calls.

$X \sim \text{Geometric}(0.45) \qquad P(X>4) = (0.55)^4 \approx 0.092$

(b) The probability that none of the first 7 calls are scam calls, given that none of the first 4 calls are scam calls.

$$P(X>7 \mid X>4) = \frac{P(X>7 \text{ and } X>4)}{P(X>4)} = \frac{P(X>7)}{P(X>4)} = \frac{(0.55)^7}{(0.55)^4} = \underbrace{(0.55)^3}_{\text{This is } P(X>3)} \approx 0.166$$

(c) The probability that none of the first $4+k$ calls are scam calls, given that none of the first 4 calls are scam calls.

$$P(X>4+k \mid X>4) = \frac{P(X>4+k)}{P(X>4)} = \frac{(0.55)^{4+k}}{(0.55)^4} = (0.55)^k = P(X>k)$$

2. Generalize your answer from problem 1(c) above. What property does this suggest for geometric random variables?

For $X \sim \text{Geometric}(p)$ and integers $0 < s < t$,
$P(X>t \mid X>s) = P(X>t-s)$.

Memoryless Property

3. Let $X \sim \text{Geometric}(p)$.

(a) Compute the mgf $M_X(t)$.

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} \underbrace{(1-p)^{x-1}p}_{P(X=x)}$$

(b) Use the infinite geometric sum formula $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ to write $M_X(t)$ without a summation.

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx}(1-p)^{x-1}p = \frac{p}{1-p}\sum_{x=1}^{\infty} e^{tx}(1-p)^x$$

$$= \frac{p}{1-p} \underbrace{\sum_{x=1}^{\infty} \left(e^t(1-p)\right)^x}_{\text{geometric series: } a=r=e^t(1-p)}$$

$$= \frac{p}{\cancel{1-p}} \cdot \frac{e^t\cancel{(1-p)}}{1-e^t(1-p)} = \frac{pe^t}{1-e^t(1-p)}$$

4. Suppose random variable X has probability mass function $P(X=x) = \frac{27}{40}\left(\frac{1}{3}\right)^x$, for integers $0 \le x \le 3$.

(a) Verify that this is a valid probability mass function.

$$\boxed{a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{1-r^n}{1-r}}$$

- nonnegative probabilities
- sum is: $\frac{27}{40}\underbrace{\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}\right)}_{\text{4 terms of a geometric series with } r=\frac{1}{3}} = \frac{27}{40}\left(\frac{1-\left(\frac{1}{3}\right)^4}{1-\frac{1}{3}}\right) = \frac{27}{40}\left(\frac{\frac{80}{81}}{\frac{2}{3}}\right) = \frac{27}{40}\cdot\frac{40}{27} = 1$

(b) Compute $E(X)$.

$$E(X) = 0 + 1\cdot\frac{9}{40} + 2\cdot\frac{3}{40} + 3\cdot\frac{1}{40} = \frac{9+6+3}{40} = \frac{9}{20}$$

(c) Find the moment generating function $M_X(t)$ of X.

$$M_X(t) = E\left(e^{tX}\right) = e^0\cdot\frac{27}{40} + e^t\cdot\frac{9}{40} + e^{2t}\cdot\frac{3}{40} + e^{3t}\cdot\frac{1}{40}$$

(d) Use the finite geometric sum formula $\sum_{n=0}^{m} ar^n = \frac{a(1-r^{m+1})}{1-r}$ to write the mgf without a summation.

$$M_X(t) = E\left(e^{tX}\right) = \underbrace{e^0\cdot\frac{27}{40} + e^t\cdot\frac{9}{40} + e^{2t}\cdot\frac{3}{40} + e^{3t}\cdot\frac{1}{40}}_{\text{common ratio: } r=\frac{e^t}{3}} = \frac{27}{40}\cdot\frac{1-\left(\frac{e^t}{3}\right)^4}{1-\left(\frac{e^t}{3}\right)} = \frac{81-e^{4t}}{40(3-e^t)}$$

(e) Use technology to compute $M_X'(0)$. Does your answer agree with the expected value of X computed directly from part (b)? — Yes!

$$M_X(t) = \frac{81-e^{4t}}{40(3-e^t)}$$

$$M_X'(0) = \frac{9}{20} = E(X)$$

MATHEMATICA:

Define the moment generating function $M_X(t)$:

```
In[1]:= mx[t_] := 27/40 ((1 - (Exp[t] / 3)^4) / (1 - Exp[t] / 3))
```

Differentiate and evaluate at $t = 0$. This gives $E(X)$.

```
In[2]:= mx'[0]
Out[2]= 9/20
```

We stopped here in class.

The remaining problems are presented as additional examples.

5. The *skewness coefficient* of the distribution of a random variable X is

$$\gamma = \frac{E[(X-\mu)^3]}{\sigma^3}$$

The skewness is 0 if the distribution is symmetric, positive if the distribution is skewed right, or negative if the distribution is skewed left.

(a) Expand $(X-\mu)^3$ and use this to express $E[(X-\mu)^3]$ in terms of the moments $E(X)$, $E(X^2)$, and $E(X^3)$. Then express γ in terms of these moments.

First: $(X-\mu)^3 = X^3 - 3X^2\mu + 3X\mu^2 - \mu^3$

Then:
$$E[(X-\mu)^3] = E(X^3) - 3E(X^2)\mu + 3E(X)\mu^2 - \mu^3$$
$$= E(X^3) - 3E(X^2)E(X) + 3E(X)E(X)^2 - E(X)^3$$
$$= E(X^3) - 3E(X^2)E(X) + 2E(X)^3$$

Also: $\sigma = \sqrt{E(X^2) - E(X)^2}$

Therefore: $$\gamma = \frac{E(X^3) - 3E(X^2)E(X) + 2E(X)^3}{\left(E(X^2) - E(X)^2\right)^{\frac{3}{2}}}$$

(b) For each of the following random variables, use Mathematica to compute the first three moments from the mgf. Then compute the skewness coefficient. Does the skewness coefficient agree with what you know about the shape of the distribution?

- $X \sim \text{Bin}\left(10, \frac{1}{2}\right)$

```
In[1]:= mx[t_] := (1/2 + 1/2 Exp[t])^10
        (mx'''[0] - 3 mx''[0] * mx'[0] + 2 (mx'[0])^3)
Out[2]= 0
```

← This is the numerator of γ.

$\gamma = 0$, which means the distribution is symmetric

- $X \sim \text{Bin}\left(10, \frac{3}{4}\right)$

```
In[3]:= mx[t_] := (1/4 + 3/4 Exp[t])^10
        (mx'''[0] - 3 mx''[0] * mx'[0] + 2 (mx'[0])^3) / (mx''[0] - mx'[0]^2)^(3/2)
Out[4]= -Sqrt[2/15]
```

$\gamma = -\sqrt{\frac{2}{15}} < 0$, so the distribution is skewed left

- $X \sim \text{Geometric}\left(\frac{1}{3}\right)$

```
In[5]:= mx[t_] := Exp[t] / (3 - 2 Exp[t])
        (mx'''[0] - 3 mx''[0] * mx'[0] + 2 (mx'[0])^3) / (mx''[0] - mx'[0]^2)^(3/2)
```

Out[6]= $\frac{5}{\sqrt{6}}$

$\gamma = \frac{5}{\sqrt{6}} > 0$, so the distribution is skewed right

- $X \sim \text{Poisson}(4)$

```
In[7]:= mx[t_] := Exp[4 (Exp[t] - 1)]
        (mx'''[0] - 3 mx''[0] * mx'[0] + 2 (mx'[0])^3) / (mx''[0] - mx'[0]^2)^(3/2)
```

Out[8]= $\frac{1}{2}$

$\gamma = \frac{1}{2} > 0$, so the distribution is skewed right

6. The monthly amount of time X (in hours) during which a manufacturing plant is inoperative due to equipment failures or power outages follows approximately a distribution with the following moment generating function:

$M_X(t) = \left(\frac{1}{1 - 7.5t}\right)^2 = (1-7.5t)^{-2}$

The amount of loss in profit due to the plant being inoperative is given by $Y = 12X + 1.25X^2$. Determine the variance of the loss in profit.

$M_X'(t) = 15(1-7.5t)^{-3}$ $\qquad M_X'(0) = 15 = E(X)$

$M_X''(t) = 337.5(1-7.5t)^{-4}$ $\qquad M_X''(0) = 337.5 = E(X^2)$

$M_X^{(3)}(t) = 10125(1-7.5t)^{-5}$ $\qquad M_X^{(3)}(0) = 10125 = E(X^3)$

$M_X^{(4)}(t) = 379687.5(1-7.5t)^{-6}$ $\qquad M_X^{(4)}(0) = 379687.5 = E(X^4)$

$E(Y) = 12E(X) + 1.25E(X^2) = 12(15) + 1.25(337.5) = 601.875$

$E(Y^2) = E(144X^2 + 30X^3 + \frac{25}{16}X^4) = 144(337.5) + 30(10125) + \frac{25}{16}(379687.5) = 945612$

$Var(Y) = E(Y^2) - E(Y)^2 = 945612 - (601.875)^2 = 583358$