Problems from last time (Poisson distribution)

3. Suppose that a machine produces items, 2% of which are defective. Let X be the number of defective items among 500 randomly-selected items produced by the machine.

(a) What is the distribution of X?

$$X \sim \text{Bin}(500, 0.02)$$

(b) What are the mean and variance of X?

$$E(X) = \underset{n}{500}(\underset{p}{0.02}) = 10, \qquad \text{Var}(X) = \underset{n}{500}(\underset{p}{0.02})(\underset{1-p}{0.98}) = 9.8$$

(c) What is $P(X = 12)$?

$$P(X=12) = \binom{500}{12}(0.02)^{12}(0.98)^{488} = 0.0955$$

[If n is big (say $n \geq 100$) and p is small (say $np \leq 10$) then $\text{Bin}(n,p)$ is well-approximated by $\text{Poisson}(np)$.]

(d) What Poisson distribution approximates the distribution of X?

$\text{Bin}(500, 0.02)$ can be approximated by $\text{Poisson}(10)$.

(e) Use your Poisson distribution to approximate $P(X = 12)$?

Let $Y \sim \text{Poisson}(10)$.

Then $P(X=12) \approx P(Y=12) = e^{-10}\frac{10^{12}}{12!} \approx 0.0948$

← This is close to the answer in part (c).

4. Let $X \sim \text{Poission}(\mu)$. Show that $P(X = k)$ increaess monotonically and then decreases monotonically as k increases, reaching its maximum when k is the largest integer less than or equal to μ.

Consider the ratio of probabilities of consecutive values k and $k-1$:

$$\frac{P(X=k)}{P(X=k-1)} = \frac{e^{-\mu}\frac{\mu^k}{k!}}{e^{-\mu}\frac{\mu^{k-1}}{(k-1)!}} = \frac{\mu^k \; k!}{\mu^{k-1}\;(k-1)!} = \frac{\mu}{k}$$

If $k < \mu$, then $\frac{\mu}{k} > 1$, so $P(X=k) > P(X=k-1)$, and the sequence of probabilities increases.

If $k = \mu$ (only possible if μ is an integer), then $\frac{\mu}{k} = 1$, so $P(X=k) = P(X=k-1)$.

If $k > \mu$, then $\frac{\mu}{k} < 1$, so $P(X=k) < P(X=k-1)$, and the sequence of probabilities decreases.

Thus, the max value of $P(X=k)$ occurs when k is the largest integer less than or equal to μ. The sequence of probabilities increases up to this value and decreases afterward.

Math 262

Review for Exam 1 Day 12

1. Suppose you roll five (fair, 6-sided) dice. Define the following events:
   A: exactly four of the five dice show the value 1
   B: exactly three of the five dice show the value 1
   C: exactly two of the five dice show the value 1
   D: the sum of the values on the five dice is 8

   (a) What is $P(A)$?

   (b) What is $P(A \cup B \cup C)$?

   (c) What is $P(A \mid D)$?

   First, note that there are 6^5 possible outcomes.

   (a) There are $\binom{5}{4}$ ways to roll four 1s, and 5 numbers for the remaining dice, so

   $$P(A) = \frac{\binom{5}{4} \cdot 5}{6^5} = \frac{25}{6^5} \approx 0.0032.$$

   (b) Similarly, event B occurs in $\binom{5}{3} \cdot 5^2$ ways, and event C occurs in $\binom{5}{2} \cdot 5^3$ ways. Since events A, B, and C are disjoint, we have

   $$P(A \cup B \cup C) = P(A) + P(B) + P(C) = \frac{\binom{5}{4} \cdot 5}{6^5} + \frac{\binom{5}{3} \cdot 5^2}{6^5} + \frac{\binom{5}{2} \cdot 5^3}{6^5} = \frac{1525}{6^5} \approx 0.1961.$$

   (c) The event $A \cap D$ occurs exactly when four of the dice show 1, and the remaining dice shows 4. Furthermore, if D occurs, then exactly one of the events A, B, or C occurs; thus $P(D)$ can be computed by the Law of Total Probability. By the definition of conditional probability,

   $$P(A \mid D) = \frac{P(A \cap D)}{P(D)} = \frac{\frac{5}{6^5}}{\frac{35}{6^5}} = \frac{1}{7} \approx 0.1428.$$

2. Consider a 20-sided die (an icosahedron) with values from 1 to 20. Roll the die one time. Let A denote the event that the value is even. Let B denote the event that the value is 13 or higher.

   (a) Are A and B disjoint events? Why?

   (b) Are A and B independent events? Why?

   (c) Calculate $P(A \cup B)$ using the inclusion-exclusion formula.

   (a) Events A and B are *not* disjoint because they have some outcomes in common (specifically, 14, 16, 18, 20).

   (b) Note that $P(A) = \frac{1}{2}$, $P(B) = \frac{8}{20}$, and $P(A \cap B) = \frac{4}{20}$. Then $P(A \cap B) = P(A)P(B)$, so events A and B are independent.

   (c) Using the inclusion-exclusion formula:

   $$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{4}{10} - \frac{2}{10} = 0.7.$$

3. An urn contains 10 balls: 4 red and 6 blue. A second urn contains 16 red balls and an unknown number of blue balls. A single ball is drawn from each urn. The probability that both balls are the same color is 0.44. How many blue balls are in the second urn?*

   Let x be the number of blue balls in urn 2; then:

   $$0.44 = P(\text{both red}) + P(\text{both blue}) = \frac{4}{10} \cdot \frac{16}{x+16} + \frac{6}{10} \cdot \frac{x}{x+16} = \frac{64+6x}{10(x+16)}$$

   Solving for x, we find $x = 4$.

*Actuary Exam P practice problem

4. Let X be the amount of time until the next message appears on your social media feed. Suppose $E(X) = 26$ seconds and $\sigma_X = 4$ seconds.

   (a) Find a lower bound on the probability that X is between 20 and 32 seconds.

   (b) Find an interval that contains X with a probability of at least 0.9.

   (a) For $P(20 < X < 30)$, use Chebyshev's inequality with $k = 1.5$:

   $$P\left(|X - 26| \geq 1.5(4)\right) \leq \frac{1}{1.5^2} = \frac{4}{9}.$$

   Taking the complement, we find that $P(20 < X < 30) \geq \frac{5}{9}$.

   (b) To find an interval that contains X with probability at least 0.9, use Chebyshev's inequality with $k = \sqrt{10}$:

   $$P\left(|X - 26| \geq 4\sqrt{10}\right) \leq \frac{1}{10}$$

   Taking the complement, we find that $P(26 - 4\sqrt{10} < X < 26 + 4\sqrt{10}) \geq 0.9$, so one such interval is $(13.35, 38.65)$.

5. An urn contains 5 red, 6 blue, and 8 green balls. If a set of 3 balls is randomly selected, what is the probability that each of the balls will be (a) of the same color; (b) of different colors? Repeat under the assumption that the balls are sampled with replacement: whenever a ball is selected, its color is noted and it is replaced in the urn before the next selection. (*Hint*: When sampling with replacement, each *ordered* selection is equally likely.)

   Sampling without replacement:

   (a) If the 3 balls are of the same color, then either they are all red, or all blue, or all green (three mutually exclusive options). Thus,

   $$P(\text{same color}) = \frac{\binom{5}{3} + \binom{6}{3} + \binom{8}{3}}{\binom{19}{3}} = \frac{86}{969} \approx 0.089.$$

   (b) To choose three balls of different colors, we must choose 1 of 5 red balls, and 1 of 6 blue balls, and 1 of 8 green balls. Thus,

   $$P(\text{different colors}) = \frac{5 \cdot 6 \cdot 8}{\binom{19}{3}} = \frac{240}{969} \approx 0.248.$$

   Sampling with replacement:

   (a) There are now 5^3 ways to choose 3 red balls, and similarly for the other colors. Thus,

   $$P(\text{same color}) = \frac{5^3 + 6^3 + 8^3}{19^3} = \frac{853}{6859} \approx 0.124.$$

   (b) There are $5 \cdot 6 \cdot 8$ combinations of 3 balls, one of each color, and each combination may be ordered in 3! ways. Thus,

   $$P(\text{different colors}) = \frac{(5 \cdot 6 \cdot 8) \cdot 3!}{19^3} = \frac{1440}{6859} \approx 0.210.$$

6. A roulette wheel has 12 numbers colored red (R) or black (B) as follows:

1	2	3	4	5	6	7	8	9	10	11	12
R	R	B	R	B	B	B	B	R	B	R	R

   Let A be the event that a spin of the wheel yields an red number. Let B be the event that a spin of the wheel yields an even number. Let C be the event that a spin of the wheel yields a number less than 7. Are events A, B, and C (pairwise) independent? Are they mutually independent?

   The following probabilities are readily obtained from the given information:

   $$P(A) = \frac{1}{2} \qquad P(B) = \frac{1}{2} \qquad P(C) = \frac{1}{2}$$

   $$P(A \cap B) = \frac{1}{4} \qquad P(A \cap C) = \frac{1}{4} \qquad P(B \cap C) = \frac{1}{4}$$

   $$P(A \cap B \cap C) = \frac{1}{6}$$

   Since $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$, $P(B \cap C) = P(B)P(C)$, the events are (pairwise) independent.

   However, $P(A \cap B \cap C) \neq P(A)P(B)P(C)$, so the events are not mutually independent.