ORDER STATISTICS

Let $X_1, \dots, X_n$ be iid continuous rvs with pdf $f(x)$ and cdf $F(x)$.

The ORDER STATISTICS are $Y_1, \dots, Y_n$, where:

$Y_1 = \min(X_1, \dots, X_n)$

Y_i is the i^{th} smallest among the values of $X_1, \dots, X_n$

$Y_n = \max(X_1, \dots, X_n)$

What is the density of Y_i?

For Y_n: use the cdf method:

Let $G_n(y)$ be the cdf of Y_n, the max of the X_i.

$$\begin{aligned} G_n(y) = P(Y_n \le y) &= P(X_1 \le y \text{ AND } X_2 \le y \text{ AND} \cdots \text{AND } X_n \le y) \\ &= P(X_1 \le y)\, P(X_2 \le y) \cdots P(X_n \le y) \\ &= F(y)\, F(y) \cdots F(y) \\ &= [F(y)]^n \end{aligned}$$

Then the pdf of Y is:

$$\frac{d}{dy} G_n(y) = \frac{d}{dy}[F(y)]^n = \boxed{n \cdot F(y)^{n-1} f(y)}$$

pdf of Y_n

power rule and chain rule

1. Let $X_1, X_2, \dots, X_n$ be iid continuous random variables with pdf $f(x)$ and cdf $F(x)$. Let $Y_1 = \min\{X_1, X_2, \dots, X_n\}$. Use the following steps to find the pdf of Y_1.

(a) Express $P(X_1 > y \cap X_2 > y \cap \cdots \cap X_n > y)$ in terms of $F(y)$.

$$\begin{aligned} \underbrace{P(X_1 > y \cap X_2 > y \cap \cdots \cap X_n > y)}_{P(Y_1 > y)} &= P(X_1 > y)\, P(X_2 > y) \cdots P(X_n > y) \\ &= (1 - F(y))(1 - F(y)) \cdots (1 - F(y)) \\ &= (1 - F(y))^n \end{aligned}$$

(b) Use your answer to part (a) to obtain an expression for $P(Y_1 < y)$, the cdf of Y_1.

$$P(Y_1 < y) = 1 - P(Y_1 > y) = 1 - (1 - F(y))^n$$

(c) Differentiate to obtain the pdf of Y_1.

$$g_1(y) = \frac{d}{dy}\left(1 - (1-F(y))^n\right) = -n(1-F(y))^{n-1}(-f(y)) = n(1-F(y))^{n-1}f(y)$$

IN GENERAL: The pdf of the ith order statistic Y_i is:

$$\boxed{g_i(y) = \frac{n!}{(i-1)!\,(n-i)!}[F(y)]^{i-1}[1-F(y)]^{n-i}f(y)}$$

2. Let X_1 and X_2 be iid Exp$\left(\frac{1}{10}\right)$. $\longrightarrow$ cdf $F(x) = 1 - e^{-x/10}$ for $x>0$, pdf $f(x) = \frac{1}{10}e^{-x/10}$ for $x>0$.

(a) What is the pdf of $Y_1 = \min(X_1, X_2)$?

$$g_1(y) = n[1-F(y)]^{n-1}f(y) = 2[1-(1-e^{-y/10})]^1\left(\tfrac{1}{10}e^{-y/10}\right) = 2[e^{-y/10}]\left(\tfrac{1}{10}e^{-y/10}\right)$$

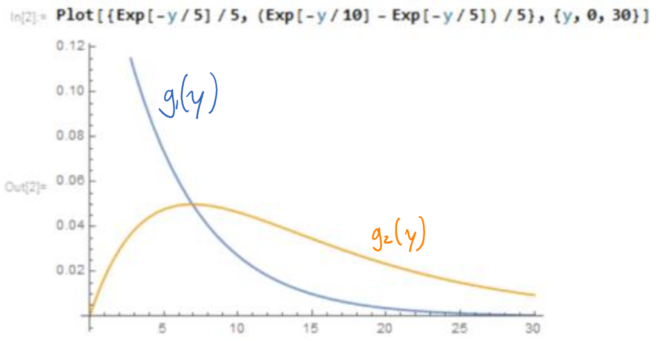
$$= \boxed{\tfrac{1}{5}e^{-y/5} \quad \text{for } y>0.}$$

exponential pdf with $\lambda = \frac{1}{5}$

(b) What is the expected value of Y_1?

$$E(Y_1) = 5$$

(c) What is the pdf of $Y_2 = \max(X_1, X_2)$? What is $E(Y_2)$?



$$g_2(y) = n[F(y)]^{n-1}f(y)$$

$$= 2[1 - e^{-y/10}]^1\left(\tfrac{1}{10}e^{-y/10}\right)$$

$$= \tfrac{1}{5}\left(e^{-y/10} - e^{-y/5}\right) \quad \text{for } y>0$$

$$E(Y_2) = \int_0^\infty y \cdot \tfrac{1}{5}\left(e^{-y/10} - e^{-y/5}\right)dy = 15$$

3. Let X_1, X_2, X_3 be iid Exp$\left(\frac{1}{10}\right)$. What is the expected value of the sample median?

Sample median: Y_2

$$g_2(y) = \frac{3!}{1!\,1!}\left[1-e^{-y/10}\right]^1\left[e^{-y/10}\right]^1\left(\frac{1}{10}e^{-y/10}\right) = \frac{6}{10}\left[1-e^{-y/10}\right]e^{-y/5}$$

$$= \frac{3}{5}\left(e^{-y/5} - e^{-3y/10}\right) \quad \text{for } y>0$$

$$E(Y_2) = \int_0^\infty y \cdot \frac{3}{5}\left(e^{-y/5} - e^{-3y/10}\right) dy = \frac{25}{3}$$

4. Let $X_1, \ldots, X_8$ be iid Unif[0,1]. Sketch the graphs of the pdfs of all eight order statistics on one plot.

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In[16]:= n = 8;
         pdfs = Table[n! / ((i - 1)! (n - i)!) y^(i - 1) (1 - y)^(n - i), {i, 1, n}]
Out[17]= {8 (1 - y)^7, 56 (1 - y)^6 y, 168 (1 - y)^5 y^2, 280 (1 - y)^4 y^3, 280 (1 - y)^3 y^4, 168 (1 - y)^2 y^5, 56 (1 - y) y^6, 8 y^7}

In[18]:= Plot[pdfs, {y, 0, 1}]
Out[18]=
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