independent, identically distributed

1. Let $X_1, X_2, \ldots, X_{300}$ be iid random variables with mean μ_X and standard deviation σ_X. Also let total $T = X_1 + X_2 + \cdots + X_{300}$ and mean $\bar{X} = \frac{T}{300}$.

(a) What are μ_T, σ_T, $\mu_{\bar{X}}$, and $\sigma_{\bar{X}}$?

Expected value is linear!

$$\mu_T = E(T) = E(X_1 + \cdots + X_{300}) = E(X_1) + \cdots + E(X_{300}) = \mu_X + \cdots + \mu_X = 300\,\mu_X$$

$$Var(T) = Var(X_1) + \cdots + Var(X_{300}) = 300\,\sigma_X^2$$ (Independent rvs)

$$\sigma_T = \sigma_X \sqrt{300}$$

$$E(\bar{X}) = E\left(\frac{T}{300}\right) = \frac{1}{300} E(T) = \mu_X$$

$$Var(\bar{X}) = Var\left(\frac{T}{300}\right) = \frac{1}{300^2} Var(T) = \frac{\sigma_X^2}{300} \quad \text{So} \quad \sigma_{\bar{X}} = \frac{\sigma_X}{\sqrt{300}}$$

(b) What distributions are good approximations for T and $\bar{X}$?

Normal! T is approx $N(300\mu_X, \sigma_X\sqrt{300})$

$\bar{X}$ is approx $N\left(\mu_X, \frac{\sigma_X}{\sqrt{300}}\right)$

CENTRAL LIMIT THEOREM (CLT):

Let $X_1, X_2, \ldots, X_n$ be iid rvs with mean μ and standard deviation σ. Let $T_n = X_1 + \cdots + X_n$ and $\bar{X}_n = \frac{T_n}{n}$.

Then the distributions of T_n and $\bar{X}_n$ approach normal distributions as $n \to \infty$.

- Distribution of T_n approaches $N(n\mu, \sigma\sqrt{n})$.
- Distribution of $\bar{X}_n$ approaches $N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$.

CONVERGENCE IN DISTRIBUTION

2. A farm packs tomatoes in crates. Individual tomatoes have mean weight of 10 ounces and standard deviation of 3 ounces. Estimate the probability that a crate of 40 tomatoes weighs between 380 and 410 ounces.

T_{40} is the weight of 1 crate of 40 tomatoes

T_{40} is approximately $N(40(10),\ 3\sqrt{40}) = N(400,\ 18.97)$

Then: $P(380 < T_{40} < 410) \approx 0.555$ pnorm(410, 400, 18.97) − pnorm(380, 400, 18.97)

3. Customers at a popular restaurant are waiting to be served. Waiting times are independent and exponentially distributed with mean $1/\lambda = 10$ minutes.

(a) Use the CLT to approximate the probability that the average wait time of 50 customers is less than 12 minutes.

$n = 50$ avg. wait time: $\overline{X}_{50}$ is approx $N\left(10, \frac{10}{\sqrt{50}}\right) = N(10,\ 1.414)$

$P(\overline{X}_{50} < 12) \approx 0.921$ pnorm(12, 10, 1.414)

(b) What is the exact probability that the average wait time of the 50 customers is less than 12 minutes?

exact distribution of T_{50} is $\text{Gamma}(\alpha = 50,\ \beta = 10)$

or: distribution of $\overline{X}_n$ is $\text{Gamma}\left(\alpha = 50,\ \beta = \frac{1}{5}\right)$ ← why? mgf's!

$P(\overline{X}_n < 12) = P(T_{50} < 50(12)) = P(T_{50} < 600) = 0.916$

pgamma(600, 50, $\frac{1}{10}$)

(WEAK) LAW OF LARGE NUMBERS

If $X_1, X_2, \ldots, X_n$ are iid rvs with mean $\mu < \infty$ and

$\overline{X}_n = \dfrac{X_1 + \cdots + X_n}{n}$, then for any $\varepsilon > 0$,

$$\lim_{n \to \infty} P(|\overline{X}_n - \mu| \geq \varepsilon) = 0.$$ ← CONVERGENCE IN PROBABILITY

INTERPRETATION: The probability that $\overline{X}_n$ is far from μ goes to zero as $n \to \infty$.

STRONG LAW OF LARGE NUMBERS

For $\overline{X}_n$ as before, $P\left(\lim_{n \to \infty} \overline{X}_n = \mu\right) = 1.$ ← ALMOST SURE CONVERGENCE

4. Suppose you flip a fair coin *lots* of times. What does the Law of Large Numbers say about the numbers of heads and tails you will observe?

to be continued...

5. Suppose that a certain casino game costs \$1 to play, and the expected winnings per game are \$0.98. What does the Law of Large Numbers say about your winnings if you play the game lots of times?