LAST TIME: If X, Y are independent, then $E(XY) = E(X)E(Y)$ and thus $\underbrace{\text{Cov}(X,Y) = 0}$ → X, Y are uncorrelated

In fact, if X and Y are independent, and $h(x,y) = g_1(x)\,g_2(y)$,
then $E(h(X,Y)) = E(g_1(X))\,E(g_2(Y))$.

NOTE: It might be that $E(XY) = E(X)E(Y)$ for dependent random variables X and Y.

1. Let $X \sim \text{Unif}[-1,1]$ and $Y = X^2$.

(a) Compute $E(X)$ and $E(XY)$. Verify that $E(XY) = E(X)E(Y)$.

symmetric bounds

$E(X) = \frac{-1+1}{2} = 0$ and $E(XY) = E(X^3) = \int_{-1}^{1} \underbrace{x^3 \tfrac{1}{2}}_{\text{odd function}} dx = 0$

$\underbrace{E(XY) = 0} = $ and $\underbrace{E(X)E(Y) = 0}$ so, X and Y are uncorrelated

(b) Are X and Y independent? Why or why not?

No, X and Y are dependent, since the value of X determines the value of Y

ANOTHER EXAMPLE OF DEPENDENT, UNCORRELATED RVS:

$U \sim \text{Unif}[0, 2\pi]$, $X = \cos(U)$ and $Y = \sin(U)$

LINEAR COMBINATIONS OF RANDOM VARIABLES

The $n=2$ case is most important:

$$E(a_1X_1 + a_2X_2 + b) = a_1E(X_1) + a_2E(X_2) + b$$ ← linearity of expected value

$$\text{Var}(a_1X_1 + a_2X_2 + b) = a_1^2\text{Var}(X_1) + a_2^2\text{Var}(X_2) + 2a_1a_2\text{Cov}(X_1, X_2)$$ ← variance is not linear!

2. Let X_1 and X_2 be the numbers that appear on rolls of two standard, fair dice.

(a) What are $E(X_i)$ and $\mathrm{Var}(X_i)$?

$$E(X_i) = \frac{1+2+3+4+5+6}{6} = \frac{7}{2}$$

$$E(X_i^2) = \frac{1^2+2^2+3^2+4^2+5^2+6^2}{6} = \frac{91}{6}$$

$$\mathrm{Var}(X_i) = E(X_i^2) - E(X_i)^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

(b) What and $E(X_1 + X_2)$ and $\mathrm{Var}(X_1 + X_2)$?

$$E(X_1 + X_2) = E(X_1) + E(X_2) = 7$$

Since X_1 and X_2 are independent,

$$\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) = \frac{35}{6}$$

3. An urn contains balls labeled 1, 2, 3, 4, 5, 6. Let Y_1 and Y_2 be the numbers on two balls drawn without replacement from the urn.

(a) What are $E(Y_i)$ and $\mathrm{Var}(Y_i)$?

$$E(Y_i) = \frac{7}{2} \quad \text{and} \quad \mathrm{Var}(Y_i) = \frac{35}{12}$$

All possible products

Y_2 \ Y_1	1	2	3	4	5	6
1	~~1~~	2	3	4	5	6
2	2	~~4~~	6	8	10	12
3	3	6	~~9~~	12	15	18
4	4	8	12	~~16~~	20	24
5	5	10	15	20	~~25~~	30
6	6	12	18	24	30	~~36~~

15 products, each equally likely

(b) What and $E(Y_1 + Y_2)$ and $\mathrm{Var}(Y_1 + Y_2)$?

$$E(Y_1 + Y_2) = E(Y_1) + E(Y_2) = 7$$

$$\mathrm{Var}(Y_1 + Y_2) = \mathrm{Var}(Y_1) + \mathrm{Var}(Y_2) + 2\,\mathrm{Cov}(Y_1, Y_2)$$
$$= \frac{35}{12} + \frac{35}{12} + 2\left(\frac{-7}{12}\right)$$
$$= \frac{14}{3}$$

$$\mathrm{Cov}(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2)$$
$$= \frac{35}{3} - \left(\frac{7}{2}\right)\left(\frac{7}{2}\right)$$

$$E(Y_1 Y_2) = \frac{1}{15}(2+3+4+5+6+6+8+10+12+15+18+20+24+30)$$
$$= \frac{175}{15} = \frac{35}{3}$$

—We stopped here in class, but below are answers to #4–6.

4. Generalize problem 2 to rolls of n dice. That is, find $E(X_1 + \cdots + X_n)$ and $\mathrm{Var}(X_1 + \cdots + X_n)$.

$$E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n) = \frac{7n}{2}$$

by independence, $\mathrm{Var}(X_1 + \cdots + X_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n) = \dfrac{35n}{12}$

5. Similarly, extend problem 3 to choosing n balls from the urn.

As before, $E(X_1 + \cdots + X_n) = \frac{7n}{2}$ but only for $n \in \{1, 2, \dots, 6\}$

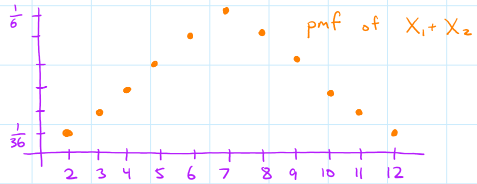
However, now $\mathrm{Var}(X_1 + \cdots + X_n) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2 \sum_{i<j} \mathrm{Cov}(X_i, X_j)$

$$= \frac{35n}{12} + 2\,\frac{n^2 - n}{2}\left(-\frac{7}{12}\right) = \frac{35n - 7n^2 + 7n}{12} = \frac{42n - 7n^2}{12} = \frac{7n(6-n)}{12} \quad \text{for } n \in \{1, \dots, 6\}$$

Note that if $n = 6$, the variance is zero.

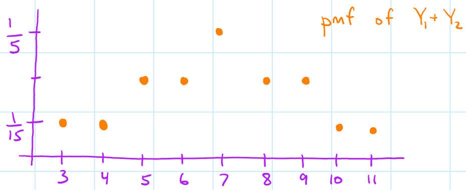
6. Find the pmfs for the sums in problems 2–5.

2 dice:



As $n \to \infty$, the pmf of $X_1 + \cdots + X_n$ becomes like a discrete version of the normal distribution.

2 balls:



... etc ...

Choose all 6 balls and the sum must be 21.