TRANSFORMATIONS OF RANDOM VARIABLES

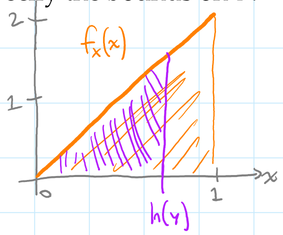
Start with a rv X with pdf $f_X(x)$, and let $Y = g(X)$.

What is the pdf of Y?

We've seen this (sort of) before:

- HW problem where $X \sim$ Poisson and $C = 150 + 5X$
- Uniform rv $X \sim \text{Unif}[0,5]$ and $Y = 3X + 2$. Guessed that Y is also unif.

1. Let X have density $f_X(x) = 2x$ for $0 \le x \le 1$, and let $Y = e^X$. What is the density of Y? Be sure to specify the bounds on Y.



Note: Y takes values in $[1, e]$

Find the cdf of Y:

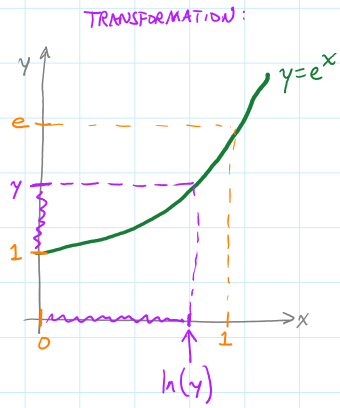
$$F_Y(y) = P(Y \le y) = P(e^X \le y)$$

$$= P(X \le \ln(y))$$

$$= \int_0^{\ln(y)} 2x\,dx = x^2 \Big|_0^{\ln(y)}$$

$$F_Y(y) = (\ln(y))^2$$

TRANSFORMATION:



Differentiate F_Y to obtain f_Y:

pdf: $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}(\ln(y))^2 = 2\ln(y)\frac{1}{y} = \boxed{\frac{2}{y}\ln(y) \quad \text{for } 1 \le y \le e}$

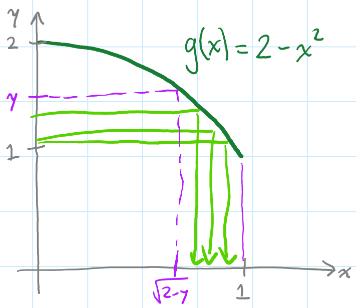
TRANSFORMATION THEOREM: Let X have density $f_X(x)$ and $Y = g(X)$, where g is strictly monotonic (incr. or decr.) on the possible values of X. Then g has an inverse function h, so $X = h(Y)$. If h is differentiable,

$$f_Y(y) = f_X(h(y))\,|h'(y)|.$$

In the previous problem, $g(x) = e^x$, which is strictly increasing and has inverse $h(y) = \ln(y)$, which is differentiable.

Thus: $f_Y(y) = f_X(h(y))\,|h'(y)| = 2(\ln(y))\left|\frac{1}{y}\right| = \frac{2}{y}\ln(y)$ for $1 \le y \le e$.

2. Let X have density $f_X(x) = 2x$ for $0 \le x \le 1$, and let $Y = 2 - X^2$. What is the density of Y?



$g(x)$ is strictly decreasing, and has inverse $h(y) = \sqrt{2-y}$, which is differentiable on $y \in [1,2]$.

By the transformation theorem:

$$f_Y(y) = f_X(h(y))\,|h'(y)| = 2\sqrt{2-y}\left|\frac{-1}{2\sqrt{2-y}}\right| = \boxed{1 \text{ for } 1 \le y \le 2.}$$

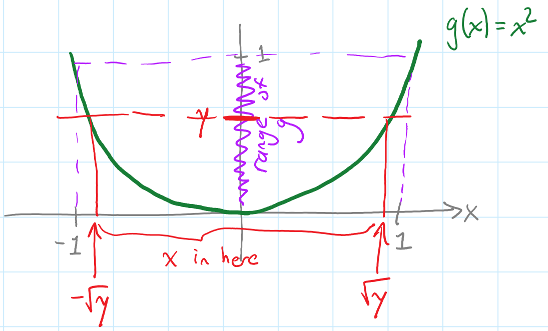
CDF method: $F_Y(y) = P(Y \le y) = P(2 - X^2 \le y) = P(X \ge \sqrt{2-y}) = \int_{\sqrt{2-y}}^{1} f_X(x)\,dx = F_X(1) - F_X(\sqrt{2-y})$

antideriv. of f_X is F_X

diff: $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(F_X(1) - F_X(\sqrt{2-y})\right) = 0 - f_X(\sqrt{2-y})\frac{-1}{2\sqrt{2-y}}$

$$= 2(\sqrt{2-y})\frac{1}{2\sqrt{2-y}} = 1 \text{ for } 1 \le y \le 2$$

3. Let X have pdf $f_X(x) = \frac{x+1}{2}$ for $-1 \le x \le 1$. Find the density of $Y = X^2$.



Find $F_Y(y) = P(Y \le y) = P(X^2 \le y)$

$$= P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{x+1}{2}\,dx$$

Can you finish this solution?