Practice Problems on Transformations of Random Variables

Math 262

1. Let X have pdf given by $f_X(x) = \frac{x+1}{2}$ for $-1 \leq x \leq 1$. Find the density of $Y = X^2$.

2. Let Y have pdf given by $f_Y(y) = 2(1-y)$ for $0 \leq y \leq 1$.

   (a) Find the density of $U_1 = 2Y - 1$.

   (b) Find the density of $U_2 = 1 - 2Y$.

   (c) Find the density of $U_3 = Y^2$.

3. Let $X \sim \text{Unif}[0,1]$. Find the density of $U = \sqrt{X}$.

4. Two sentries are sent to patrol a road that is 1 mile long. The sentries are sent to points chosen independently and uniformly along the road. Find the probability that the sentries will be less than $\frac{1}{2}$ mile apart when they reach their assigned posts.

5. The joint distribution for the lifetimes of two different types of components operating in a system is given by

$$f(y_1, y_2) = \begin{cases} \frac{1}{8} y_1 e^{-(y_1+y_2)/2} & \text{if } y_1 > 0, y_2 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

   Find the density function for the ratio $U = \frac{Y_2}{Y_1}$.

6. Suppose X and Y are independent exponential rvs with parameter λ. Find the joint density of $V = \frac{X}{Y}$ and $W = X + Y$. Use the joint density to find the marginal distributions.

7. Let X and Y have joint density $f(x, y)$. Let (R, Θ) be the polar coordinates of (X, Y).

   (a) Give a general expression for the joint density of R and Θ.

   (b) Suppose X and Y are independent with $f(x) = 2x$ for $0 < x < 1$ and $f(y) = 2y$ for $0 < y < 1$. Use your result to find the probability that (X, Y) lies inside the circle of radius 1 centered at the origin.

8. Let $X_1, X_2, \ldots, X_n$ denote a random sample from the uniform distribution on $[0, 1]$. Let Y_1 and Y_n be the smallest and largest, respectively, among the X_i. Find the pdf for the range $R = Y_n - Y_1$.

   Hint: The joint pdf for Y_1 and Y_n is $g(y_1, y_n) = n(n-1)(y_n - y_1)^{n-2}$ for $0 \leq y_1 \leq y_n \leq 1$. (See exercise 141 in Chapter 4 of Carlton and Devore.)