1. Suppose f(x) and g(x) are probability density functions. Under what conditions on the constants α and β will the function $\alpha f(x) + \beta g(x)$ be a probability density function?

Since f and g are pdfs:

$$f(x) \ge 0, \quad g(x) \ge 0, \quad \int_{-\infty}^{\infty} f(x) \, dx = 1, \quad \text{and} \quad \int_{-\infty}^{\infty} g(x) \, dx = 1$$
If $\alpha f(x) + \beta g(x)$ is a pdf, for all possible pdfs $f(x)$ and $g(x)$,
i+ must be that $\alpha \ge 0, \quad \beta \ge 0, \quad \text{and}:$

$$1 = \int_{-\infty}^{\infty} (\alpha f(x) + \beta g(x) \, dx = \alpha \int_{-\infty}^{\infty} f(x) \, dx + \beta \int_{-\infty}^{\infty} g(x) \, dx = \alpha + \beta$$
So $\alpha + \beta = 1.$

2. Let $X \sim \text{Exp}(\lambda)$, $0 \le s$ and $0 \le t$. Since X is memoryless, is it true that (X > s + t) and (X > t) are independent events?

Memoryless Property:
$$P(X > s+t | X > t) = P(X > s)$$

Since $P(X > s) \neq P(X > s+t)$, we have
 $P(X > s+t | X > t) \neq P(X > s+t)$,
so the events $X > s+t$ and $X > t$ are not independent.

3. Among 30 raffle tickets, six are winners. Felicia buys 10 tickets. Find the probability that she gets exactly three winners.

Let X be the number of winning tickets that Felicia buys.
Then X ~ Hypergeometric (n = 10, M = 6, N = 30).
Thus:
$$P(X = 3) = \frac{\binom{6}{3}\binom{74}{7}}{\binom{30}{10}} \approx 0.2304$$
 R: dhyper (3, 6, 24, 10)

4. Let *X* and *Y* be iid exponential rvs with parameter λ . Let (*R*, Θ) be the polar coordinates of (*X*, *Y*). What is the joint density of *R* and Θ ?

Joint density of X and Y: $f(x, y) = \lambda^2 e^{-\lambda(x+y)}$ for x > 0, y > 0

Note that $R = \sqrt{X^2 + Y^2}$, $\Theta = a \tan\left(\frac{Y}{X}\right)$, $X = R \cos \Theta$, $Y = R \sin \Theta$, R > 0, $O < \Theta < \frac{\pi}{2}$

The Jacobian matrix is:

$$M = \begin{bmatrix} \frac{\partial X}{\partial r} & \frac{\partial X}{\partial \Theta} \\ \frac{\partial Y}{\partial r} & \frac{\partial Y}{\partial \Theta} \end{bmatrix} = \begin{bmatrix} \cos \Theta & -r \cdot \sin \Theta \\ \sin \Theta & r \cdot \cos \Theta \end{bmatrix} \quad \text{with determinant } |M| = r \cdot \cos^2 \Theta + r \cdot \sin^2 \Theta = r$$

- By the (bivariate) Transformation Theorem, the joint density of R and Θ is: $g(r, \Theta) = f(x \cdot \cos \Theta, y \cdot \sin \Theta) |r| = \lambda^2 r e^{-\lambda r (\cos \Theta + \sin \Theta)}$ for $O < r, O < \Theta < \frac{\pi}{2}$.
- 5. Let $X_1, X_2, ..., X_{10}$ be iid random variables denoting bids on an item that is for sale in an auction. The item will be sold to the highest bidder. If the bids are independent and uniformly distributed between 10 and 30, what is the expected value of the sale price?

Each
$$X_i$$
 has $pdf = f_x(x) = \frac{1}{20}$ and $cdf = F_x(x) = \frac{x-10}{20}$ for $10 \le x \le 30$.
 $Y_{10} = \max(X_i)$ has $pdf = g_{10}(y) = 10 \left[F_x(y)\right]^q f_x(y) = 10 \left[\frac{y-10}{20}\right]^q \cdot \frac{1}{20} = \frac{(y-10)^q}{2(20)^q}$ for $10 \le y \le 30$.
Thus, $E(Y_{10}) = \int_{10}^{30} y \cdot \frac{(y-10)^q}{2(20)^q} dy = \frac{310}{11}$.

6. Suppose *B* and *C* are iid Unif[0,1]. Find the probability that the roots of the equation $x^2 + Bx + C = 0$ are real.

The roots are
$$x = \frac{-B \pm \sqrt{B^2 - 4C}}{2}$$
, which are real iff $B^2 - 4C \ge 0$,
or equivalently, $C \le \frac{B^2}{4}$.
Then: $P(\text{real roots}) = P(C \le \frac{B^2}{4})$
 $= \iint_R 1 \, dA = \text{Area}(R)$
 $= \int_0^1 \frac{b^2}{4} \, db = \frac{1}{12}$