BIVARIATE TRANSFORMATION THEOREM

Let X_1 and X_2 have joint density $f(x_1, x_2)$.

Let $Y_1 = u_1(X_1, X_2)$ and $Y_2 = u_2(X_1, X_2)$. ← Transformation

Also: $X_1 = v_1(Y_1, Y_2)$ and $X_2 = v_2(Y_1, Y_2)$. ← Inverse transformation.

Let $M = \begin{bmatrix} \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial y_2} \\ \frac{\partial v_2}{\partial y_1} & \frac{\partial v_2}{\partial y_2} \end{bmatrix}$ ← the Jacobian matrix

Then the joint density of Y_1 and Y_2 is given by

$$\boxed{g(y_1, y_2) = f(\underbrace{v_1(y_1, y_2)}_{x_1}, \underbrace{v_2(y_1, y_2)}_{x_2}) \cdot |\det(M)|}$$

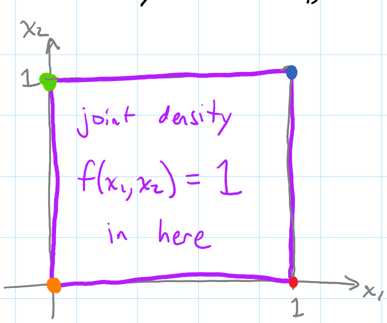
[Compare to the 1-var. transformation theorem:

$$f_Y(y) = f_X(h(y)) \cdot |h'(y)|$$
]

EXAMPLE: X_1 and X_2 are iid Unif $[0,1]$.

Let $\underset{Y_1 = u_1(x_1, x_2)}{Y_1 = X_1 + X_2}$ and $\underset{Y_2 = u_2(x_1, x_2)}{Y_2 = X_1 - X_2}$.

Joint density of X_1, X_2:



the transformation is linear:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

$(x_1, x_2) \longrightarrow (y_1, y_2)$

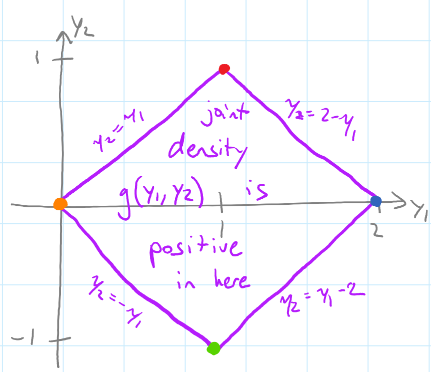
$(0,0) \longrightarrow (0,0)$

$(1,0) \longrightarrow (1,1)$

$(1,1) \longrightarrow (2,0)$

$(0,1) \longrightarrow (1,-1)$

Possible values of $Y_1\ Y_2$:



$y_2 \le y_1$

$y_2 \le 2 - y_1$

$y_2 \ge -y_1$

$y_2 \ge y_1 - 2$

INVERSE: $X_1 = v_1(Y_1, Y_2)$, $X_2 = v_2(Y_1, Y_2)$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

so $v_1(y_1,y_2) = \frac{1}{2}(y_1+y_2)$ and $v_2(y_1,y_2) = \frac{1}{2}(y_1-y_2)$

Then: $M = \begin{bmatrix} \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial y_2} \\ \frac{\partial v_2}{\partial y_1} & \frac{\partial v_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ and $\det(M) = \frac{1}{2}\left(-\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}\right) = -\frac{1}{2}$

The joint density of Y_1 and Y_2 is:

$$g(y_1,y_2) = f(v_1(y_1,y_2), v_2(y_1,y_2)) \cdot |\det(M)| = 1 \cdot \left|-\frac{1}{2}\right| = \frac{1}{2}$$

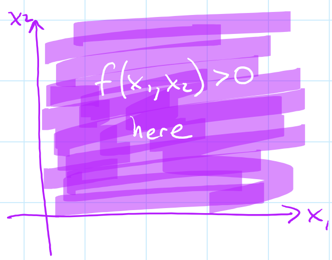
on the region found above.

1. Suppose X_1 and X_2 are independent exponential rvs with parameter λ.

exponential pdf: $\lambda e^{-\lambda x}$, $x > 0$

(a) Find the joint density of $Y_1 = \frac{X_1}{X_2}$ and $Y_2 = X_1 + X_2$.

Joint density of X_1, X_2: $f(x_1,x_2) = \lambda^2 e^{-\lambda(x_1+x_2)}$ for $x_1>0$, $x_2>0$



Invert the transformation:

$Y_1 = \frac{X_1}{X_2}$ so $X_1 = Y_1 X_2$

and then $Y_2 = X_1 + X_2 = Y_1 X_2 + X_2$,

which implies $X_2 = \frac{Y_2}{Y_1+1}$

Y_1 and Y_2 take on any pair of positive values.

Also: $X_1 = Y_1 X_2$, so $X_1 = \frac{Y_1 Y_2}{Y_1+1}$

Matrix $M = \begin{bmatrix} \frac{y_2}{(y_1+1)^2} & \frac{y_1}{y_1+1} \\ \frac{-y_2}{(y_1+1)^2} & \frac{1}{y_1+1} \end{bmatrix}$ so $\det(M) = \frac{y_2}{(y_1+1)^3} - \frac{-y_1 y_2}{(y_1+1)^3} = \frac{y_2}{(y_1+1)^2}$

Joint density of Y_1, Y_2:

$$g(y_1,y_2) = f\left(\frac{y_1 y_2}{y_1+1}, \frac{y_2}{y_1+1}\right) \cdot \left|\frac{y_2}{(y_1+1)^2}\right| = \lambda^2 e^{-\lambda\left(\frac{y_1 y_2}{y_1+1} + \frac{y_2}{y_1+1}\right)} \frac{y_2}{(y_1+1)^2} \text{ for } y_1>0,\ y_2>0$$

We stopped here in class, but the following solutions are provided as additional examples:

(b) Use the joint density to find the marginal densities of Y_1 and Y_2.

Integrate: $g_{Y_1}(y_1) = \int_0^{\infty} g(y_1, y_2)\, dy_2 = \frac{1}{(1+y_1)^2}$ and $g_{Y_2}(y_2) = \int_0^{\infty} g(y_1, y_2)\, dy_1 = \lambda^2 y_2 e^{-\lambda y_2}$.

2. Let X and Y have joint density $f(x, y)$. Let (R, Θ) be the polar coodinates of (X, Y).

(a) Give a general expression for the joint density of R and Θ.

Note that $R = \sqrt{X^2 + Y^2}$ and $\Theta = \arctan\left(\frac{Y}{X}\right)$, $X = R \cos \Theta$ and $Y = R \sin \Theta$

Jacobian determinant:

$$|M| = \begin{vmatrix} \frac{\partial}{\partial r} r \cos\theta & \frac{\partial}{\partial \theta} r \cos\theta \\ \frac{\partial}{\partial r} r \sin\theta & \frac{\partial}{\partial \theta} r \sin\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

Joint density of R and Θ:

$$g(r, \theta) = f(r\cos\theta, r\sin\theta)\, |M| = f(r\cos\theta, r\sin\theta) \cdot r$$

(b) Suppose X and Y are independent with $f(x) = 2x$ for $0 < x < 1$ and $f(y) = 2y$ for $0 < y < 1$. Use your result from part (a) to find the probability that (X, Y) lies inside the circle of radius 1 centered at the origin.

Joint density of X and Y: $f(x, y) = 4xy$ for $0 < x < 1$, $0 < y < 1$

From part (a), joint density of R and Θ is:

$$g(r, \theta) = f(r\cos\theta, r\sin\theta)r = 4(r\cos\theta)(r\sin\theta)r = 4r^3 \cos\theta \sin\theta$$

The point (X, Y) lies within the unit circle iff $R < 1$.

Since both X and Y are both positive, $0 < \Theta < \frac{\pi}{2}$, so prob. $R < 1$ is given by

$$P(R < 1) = \int_0^{\pi/2} \int_0^1 4r^3 \cos\theta \sin\theta \, dr\, d\theta = \frac{1}{2}.$$