INVESTIGATE: $\dfrac{F_n}{F_{n-1}}$

THEOREM: $\displaystyle\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2}$

proof: by definition: $F_n = F_{n-1} + F_{n-2}$

divide: $\dfrac{F_n}{F_{n-1}} = 1 + \dfrac{F_{n-2}}{F_{n-1}}$

take limit: $x \rightarrow \boxed{\displaystyle\lim_{n\to\infty} \frac{F_n}{F_{n-1}}} = 1 + \boxed{\displaystyle\lim_{n\to\infty} \frac{F_{n-2}}{F_{n-1}}} \leftarrow \frac{1}{x}$

let $x = \displaystyle\lim_{n\to\infty} \frac{F_n}{F_{n-1}}$. Then: $x = 1 + \dfrac{1}{x}$

or: $x^2 = x + 1$

$x^2 - x - 1 = 0$

solve for x: $x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \dfrac{1 \pm \sqrt{5}}{2}$

Since $F_n > 0$, $x > 0$ also, so $x = \dfrac{1+\sqrt{5}}{2}$. golden ratio

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OBSERVATIONS:

$$\lim_{n\to\infty} \frac{F_n}{F_{n-k}} = \varphi^k = \left(\frac{1+\sqrt{5}}{2}\right)^k$$

$\varphi^k = \varphi^{k-1} + \varphi^{k-2}$ because φ satisfies $\varphi^2 = \varphi + 1$

CHECKING AN IDENTITY:

$$\underbrace{F_{n-1}F_{n+1} - F_n^2}_{f(n)} = \underbrace{(-1)^n}_{g(n)}$$

$f(n) = g(n)$ for integers n

LHS
left hand side

RHS
right hand side