MATH 242: Wednesday, April 29, 2020

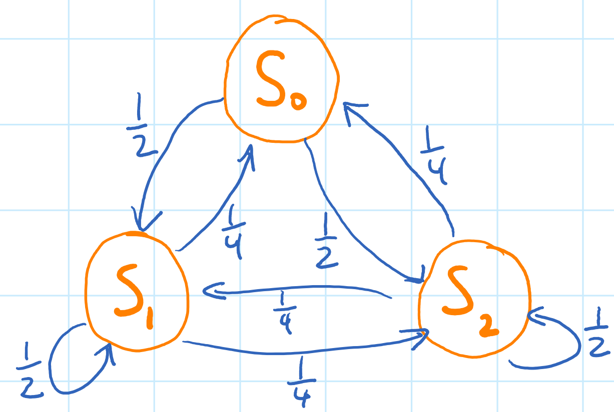
TODAY:

- Introduce Markov chains — 3 perspectives
- Markov chain inverse problem

MARKOV CHAINS

A Markov chain is a sequence of events (called "states"), where the probability of each event depends only on the previous state.

EXAMPLE:



Transition Matrix

$$P = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

Entry in row i, column j gives probability $S_j \rightarrow S_i$.

QUESTION: In the long run, what are the relative frequencies (or probabilities) of the three states?

PERSPECTIVE 1: RANDOM WALKS

Start a random walk on any state. The walk chooses its next step according to the probabilities associated with the current state. Simulate the walk for lots of steps and count the number of visits to each state.

PERSPECTIVE 2: MATRIX POWERS

Suppose initial state is 0.

Probabilities for the next state: $\begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

Probabilities for the state after that: $\begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{8} \\ \frac{3}{8} \end{bmatrix}$

Probabilities for the state after that: $P^2 \cdot \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{16} \\ \frac{13}{32} \\ \frac{13}{32} \end{bmatrix}$

Probabilities for the 100^{th} state: $P^{100} \cdot \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = ?$

PERSPECTIVE 3: EIGENVECTORS!

If $\mathbf{v}$ is the steady-state vector, then $P\mathbf{v} = \mathbf{v}$.

$\mathbf{v}$ is an eigenvector corresponding to eigenvalue $\lambda = 1$

$(P\mathbf{v} = 1\mathbf{v})$

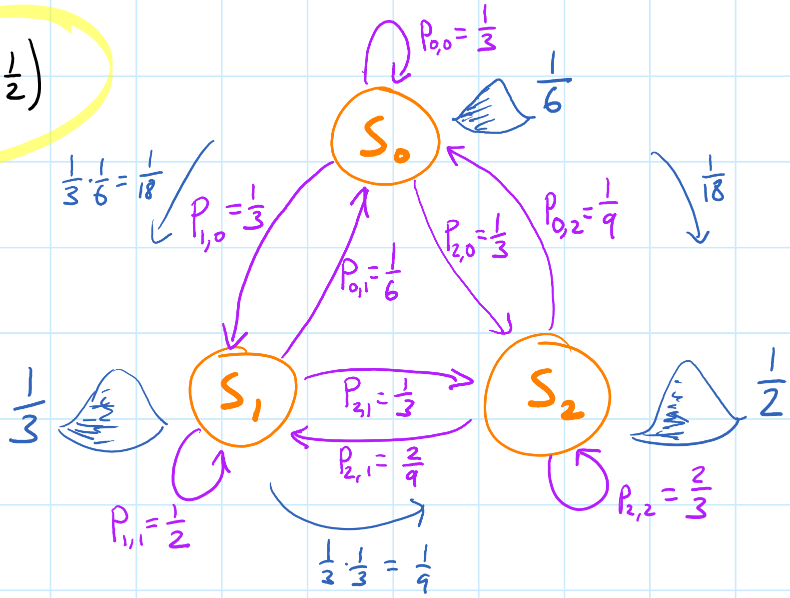
To find the steady-state vector, just look for the eigenvector corresponding to eigenvalue 1.

MARKOV CHAIN INVERSE PROBLEM

Suppose we are given a steady-state vector $\mathbf{v}$.
Find a transition matrix P with this steady-state vector.

EXAMPLE: $\mathbf{v} = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right)$

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{9} \\ \frac{1}{3} & \frac{1}{2} & \frac{2}{9} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$



ALGORITHM:

1. Select an $n \times n$ proposal transition matrix Q, of nonnegative entries, each column sums to 1.

2. Obtain the actual transition matrix P as follows:

$\mathbf{v} = (v_1, v_2, \ldots, v_n)$

Given states S_i and S_j with $i \neq j$, define transition probability $P_{i,j}$

- If $v_j \leq v_i$, then $P_{i,j} = Q_{i,j}$
- If $v_j > v_i$, then $P_{i,j} = \frac{v_i}{v_j} Q_{j,i}$

Define $P_{i,i} = 1 - \sum_{k \neq i} P_{k,i}$ ← Choose diagonal entries so that column sums are 1.

Does it really work? Check that $P\mathbf{v} = \mathbf{v}$.