MATH 242: Wednesday, April 22

TODAY:

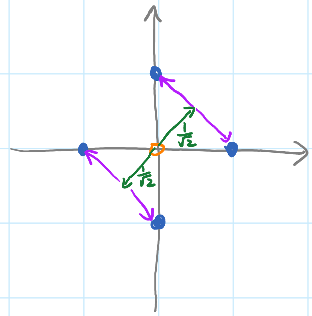
- 2D Random Walk observations
- Random walk proofs: 2D and 3D return to origin

RECALL: 1-D

Probability RW at origin at step $n = 2k$: $\sim \frac{1}{\sqrt{\pi k}}$

Expected number of returns to origin: $\approx \sum_{k=1}^{\infty} \frac{1}{\sqrt{\pi k}}$ DIVERGES

NOW: 2-D



For each step of the 2D RW:

- step distance $\frac{1}{\sqrt{2}}$ along line of slope 1 } RW1
- then step dist $\frac{1}{\sqrt{2}}$ along line of slope -1 } RW2

The 2D random walk returns to the origin iff both RW1 and RW2 return to the origin (as 1D walks) simultaneously.

So, the probability that a 2D random walk returns to the origin at step $n=2k$ is asymptotic to:

$$\underbrace{\frac{1}{\sqrt{\pi k}}}_{RW1} \cdot \underbrace{\frac{1}{\sqrt{\pi k}}}_{RW2} = \frac{1}{\pi k}$$

The expected (average) number of returns to the origin for a 2D random walk:

$$\frac{1}{\pi} + \frac{1}{2\pi} + \frac{1}{3\pi} + \cdots = \sum_{k=1}^{\infty} \frac{1}{\pi k} = \frac{1}{\pi} \boxed{\sum_{k=1}^{\infty} \frac{1}{k}}$$

← HARMONIC SERIES DIVERGES

A simple, symmetric 2D random walk returns to the origin infinitely many times.

1D and 2D random walks are said to be recurrent because they return to the origin.

WHAT ABOUT 3D?

The probability that a 3D RW returns to the origin at step $n=2k$ is: $\left(\frac{1}{\sqrt{\pi k}}\right)^3 = (\pi k)^{-3/2}$

The expected number of returns to the origin:

$$\frac{1}{\pi^{3/2}} + \frac{1}{(2\pi)^{3/2}} + \frac{1}{(3\pi)^{3/2}} + \cdots = \frac{1}{\pi^{3/2}} \boxed{\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}}$$

p-series with $p = 3/2$

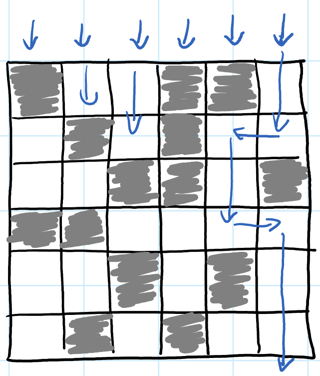
CONVERGES!

The expected number of times that a 3D random walk returns to the origin is finite.

3D (and higher-D) random walks are transient because they don't always return to the origin.

- Introduce Percolation – topic for the next several days

PERCOLATION THEORY



Start with a $n \times n$ grid of squares:

Fix a probability p.

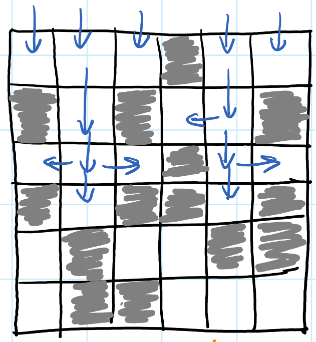
$0 \le p \le 1$

Let each square be "open" with probability p and "closed" with probability $1-p$.

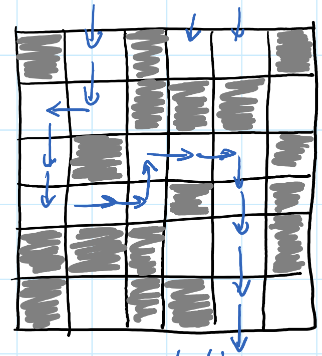
Pour water on the top of the grid.

QUESTION: Can water flow through the grid from top to bottom?

EXAMPLES:



no percolation



percolation

QUESTION: What is the probability of percolation?

How does it depend on p? ... on n?