MEAN-MEDIAN MAP: start with $\{x_1, x_2, \ldots, x_n\}$

sequence of values

Compute x_{n+1}:

$$\frac{x_1 + x_2 + \cdots + x_n + x_{n+1}}{n+1} = \text{median}(x_1, x_2, \ldots, x_n)$$

The sequence of medians is monotone.

either increasing or decreasing, not both

why?

Let $S_n = \{x_1, x_2, \ldots, x_n\}$

Compute:

$$x_1 + x_2 + \cdots + x_n + x_{n+1} = (n+1)\,\text{median}(S_n) \qquad (1)$$

$$x_1 + x_2 + \cdots + x_{n+1} + x_{n+2} = (n+2)\,\text{median}(S_{n+1}) \qquad (2)$$

Subtract (1) from (2):

$$x_{n+2} = \underbrace{(n+2)}_{(n+1)+1}\,\text{median}(S_{n+1}) - (n+1)\,\text{median}(S_n)$$

$$x_{n+2} = (n+1)\,\text{median}(S_{n+1}) + \text{median}(S_{n+1}) - (n+1)\,\text{median}(S_n)$$

$$x_{n+2} = (n+1)\Big[\underbrace{\text{median}(S_{n+1}) - \text{median}(S_n)}_{(*)}\Big] + \text{median}(S_{n+1})$$

→ If $\text{median}(S_{n+1}) \geq \text{median}(S_n)$, then $(*)$ is nonnegative, so $x_{n+2} \geq \text{median}(S_{n+1})$. Then, by def. of median, $\text{median}(S_{n+2}) \geq \text{median}(S_{n+1})$. Thus, the sequence of medians does not decrease (it's monotone increasing).

→ If $\text{median}(S_{n+1}) \leq \text{median}(S_n)$, then $(*)$ is not positive. By similar reasoning as above, we get a non-increasing sequence of medians (i.e, monotone decreasing).

Corollary: If two consecutive medians are equal, then the sequence of medians is constant. Why? Both conclusions above apply.

If the sequence of medians is constant, then the sequence of values is also constant:

$$x_{n+2} = (n+1)\underbrace{\left[\text{median}(S_{n+1}) - \text{median}(S_n)\right]}_{\text{This is } 0 \text{ if } \text{median}(S_{n+1}) = \text{median}(S_n)} + \text{median}(S_n)$$

Then $x_{n+2} = \text{median}(S_n)$.