Sequence: 0, 1, 2, 5, 12, 29, 70, ... ← Pell Numbers

$1+2(2)=5$

$2+2(5)=12$

Definition: $P_0=0$, $P_1=1$, $P_n = P_{n-2} + \boxed{2}P_{n-1}$ for $n \geq 2$

Interesting application: approximating $\sqrt{2} \approx 1.4142$

$\frac{1}{1}=1$, $\frac{3}{2}=1.5$, $\frac{7}{5}=1.4$, $\frac{17}{12}=1.41\overline{6}$, $\frac{41}{29}=1.41379, \ldots$

Note: If $\frac{x}{y}$ is one of these fractions, then $\boxed{x^2-2y^2=\pm 1}$ Pell's equation

$17^2-2(12)^2 = 289-2(144)=1$

If $x^2-2y^2=0$, then $x^2=2y^2$, so $\frac{x^2}{y^2}=2$, and thus $\frac{x}{y}=\sqrt{2}$.

There are no integers x, y such that $\frac{x}{y}=\sqrt{2}$,

so there are no integer solutions to $x^2-2y^2=0$.

However, we can find integer solutions to $x^2-2y^2=\pm 1$,

these involve the Pell numbers.

A Cool Pell Identity:

every third Pell num. P_{3n} | Cube of Pell nums P_n^3 | Pell nums P_n

0	0	0
5 =	8·1 −	3·1
70 =	8·8 +	3·2
985 =	8·125 −	3·5
13 860 =	8·1728 +	3·12
195 025 =	8·24 389 −	3·29
2 744 210 =	8·343 000 +	3·70

$8\cdot 8+3\cdot 2=64+6=70$

$8\cdot 125-3\cdot 5=1000-15=985$

$8\cdot 1728+3\cdot 12=13824+36=13860$

The pattern: $P_{3n} = 8(P_n)^3 + (-1)^n 3P_n$