RANDOM WALKS — RETURN TO ORIGIN

SIMPLE

1D: return somewhat frequently to the origin

2D: ?

3D: ?

Find the average number of times that an infinite random walk returns to the origin.

1D: Let p_n be the probability that a 1-D random walk is at the origin after $2n$ steps.

(Note that the walk can only return to the origin after an even number of steps.)

Then the average number of returns to the origin is

$$p_1 + p_2 + p_3 + \cdots = \sum_{n=1}^{\infty} p_n$$

due to additivity of expected value

How to find p_n? Count possible random walks:

$$p_n = \frac{\binom{2n}{n}}{2^{2n}}$$

BINOMIAL COEFFICIENT: gives the number of ways of choosing $2n$ items out of n items

$\binom{2n}{n} = \frac{(2n)!}{n!\,n!}$ is the number of walks of length $2n$ that return to the origin at step $2n$

Why? $2n$ steps total, choose any n to be steps to the left, the other n steps must be to the right.

2^{2n} is the total number of random walks of length $2n$

Why? Choose either of 2 directions at each step.

$$\underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{2n \text{ steps}} = 2^{2n}$$

Now apply **STERLING'S FORMULA:** $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

asymptotically equal, as $n \to \infty$

$$p_n = \frac{\binom{2n}{n}}{2^{2n}} = \frac{\frac{(2n)!}{n!\,n!}}{2^{2n}} = \frac{(2n)!}{(n!)^2\, 2^{2n}} \sim \frac{\sqrt{2\pi(2n)}\left(\frac{2n}{e}\right)^{2n}}{2\pi n \underbrace{\left(\frac{n}{e}\right)^{2n} 2^{2n}}_{=\left(\frac{2n}{e}\right)^{2n}}} = \frac{2\sqrt{\pi n}}{2\pi n} = \frac{1}{\sqrt{\pi n}}$$

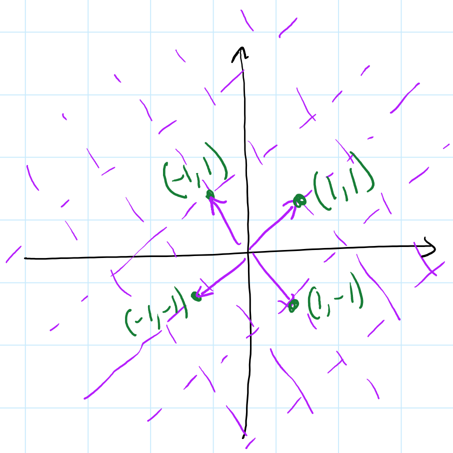
Thus: average number of returns to the origin:

$$\sum_{n=1}^{\infty} p_n \approx \sum_{n=1}^{\infty} \frac{1}{\sqrt{n\pi}} = \frac{1}{\sqrt{\pi}} \underbrace{\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}}_{\text{diverges}} = \infty$$

$\boxed{\text{This implies a 1-D random walk returns infinitely often to the origin.}}$

p-series: (Calc. II) $\displaystyle \sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \infty & \text{if } p \le 1 \\ \text{finite} & \text{if } p > 1 \end{cases}$

2D:



Suppose a 2-D random walk moves NE, NW, SE, SW with equal probability.

View this 2-D walk as two 1-D walks:

One for the x-coord.
One for the y-coord.
} independent

The 2-D walk returns to the origin exactly when both 1-D walks return to the origin at the same time.

Let q_n be the probability that the 2D walk is at the origin at step $2n$. Then the average number of returns to the origin is:

$$\sum_{n=1}^{\infty} q_n = \sum_{n=1}^{\infty} (p_n)^2 \approx \underbrace{\sum_{n=1}^{\infty} \frac{1}{\pi n}}_{\text{diverges!}} = \infty$$

Thus, the 2-D random walk also returns infinitely often to the origin. It's just harder to observe computationally than in the 1-D case because it can take a <u>lot</u> of steps to get back to the origin.