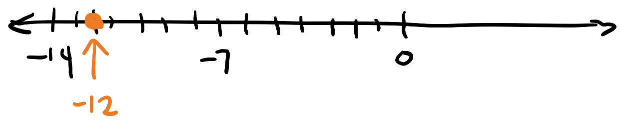
Math 234

Modular Arithmetic and $\mathbf{Z}_n$



Day 24

1. (a) What is the value of 23 (mod 7)?

$23 \equiv \boxed{2} \pmod 7$ congruent

$\{0, 1, 2, 3, 4, 5, 6\}$

(b) What is the value of $-12 \pmod 7$?

$-12 = -14 + 2$ so $-12 \equiv \boxed{2} \pmod 7$

(c) Is it true that $23 \pmod 7 = -12 \pmod 7$?

Yes, since $2 = 2$

(d) Is it true that $23 \equiv 12 \pmod 7$?

No. $2 \neq 5$

23 is 2 more than a multiple of 7, but 12 is 5 more than a multiple of 7

2. Using the facts that $46 \equiv 7 \pmod{13}$ and $17 \equiv 4 \pmod{13}$, using modular arithmetic to efficiently find an integer $0 \leq d \leq 12$ such that

multiples of 13: 13, 26, 39, 52

(a) $63 \equiv d \pmod{13}$. Note that $63 = 46 + 17$.

$63 \equiv 7 + 4 = 11 \pmod{13}$

$63 = (4)13 + 11$

(b) $29 \equiv d \pmod{13}$. Note that $29 = 46 - 17$.

$29 \equiv 7 - 4 = 3 \pmod{13}$

$29 = 2(13) + 3$

(c) $782 \equiv d \pmod{13}$. Note that $782 = 46 \times 17$.

$782 \equiv 7 \cdot 4 = 28 \equiv 2 \pmod{13}$

$782 = 13(60) + 2$
$= 780 + 2$

(d) $143 \equiv d \pmod{13}$. Note that $143 = (2 \times 46) + (3 \times 17)$.

$143 \equiv (2 \times 7) + (3 \times 4) = 14 + 12 = 26 \equiv 0 \pmod{13}$

So $143 \equiv 0 \pmod{13}$

$143 = 11 \cdot 13$

3. Find the units digit of 7^{2022}. Then do the same for 37^{2022}.

mod 10!

Find a pattern:

$7^0 = 1$

$7^1 = 7$

$7^2 = 49 \equiv 9 \pmod{10}$

$7^3 \equiv 3 \pmod{10}$

$7^4 \equiv 1 \pmod{10}$

$9 \cdot 7 = 63$

So $9 \cdot 7 \equiv 3 \pmod{10}$

$7^2 \cdot 7 \equiv 3 \pmod{10}$

$2022 = 4 \cdot 505 + 2$

$7^{2022} = 7^{4 \cdot 505 + 2} = (7^4)^{505} \cdot 7^2$

$\equiv 1^{505} \cdot 7^2 \pmod{10}$

$\equiv 1 \cdot 9$

$\boxed{7^{2022} \equiv 9 \pmod{10}}$

$f: S \times S \to S$

4. Recall that a **binary operation** on a set S is a function from $S \times S$ to S. Determine whether each of the functions below is a binary operation, and if so, identify set set S.

   (a) The logical *or* operation, as in $r \vee s$, where r and s are logical true/false values.

   $S = \{\text{true}, \text{false}\}$ $\quad \vee: S \times S \to S$ $\quad$ Yes!

   $S \times S = \{(T,T), (T,F), (F,T), (F,F)\}$

   (b) The logical *and* operation, as in $r \wedge s$, where r and s are logical true/false values.

   again, $S = \{\text{true}, \text{false}\}$ $\quad \wedge: S \times S \to S$ $\quad$ Yes!

r	s	$r \to s$
T	T	T
T	F	F
F	T	T
F	F	T

   (c) The logical implication operation, as in $r \to s$, where r and s are logical true/false values.

   Yes, also a binary operation.

   (d) The numerical less than operation, as in $r < s$, where r and s are real numbers.

   $<: \mathbb{R} \times \mathbb{R} \to \{\text{true}, \text{false}\}$ $\quad$ NOT a binary operation

5. Let $\cdot$ be the usual multiplication operation for real numbers in some set S.

   (a) If $S = \mathbf{R}$, is $\cdot$ a binary operation?

   Yes, since the product of two real numbers is a real number.

   (b) If $S = \mathbf{R}^+$, is $\cdot$ a binary operation?

   Yes

   (c) If $S = \mathbf{Z}$, is $\cdot$ a binary operation?

   Yes

   (d) If $S = \mathbf{Z}^-$ (negative integers), is $\cdot$ a binary operation?

   No, since the product of two negative integers is positive.

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6. Let $+$ be the usual addition operation on real numbers.

   (a) If $A = \{x \mid x > 0\}$, is A closed under $+$?

   (b) If $A = 2\mathbf{Z}$, the set of even integers, is A closed under $+$?

   (c) If $A = \{n \in \mathbf{Z} \mid n \text{ is odd}\}$, is A closed under $+$?

   (d) If $A = \mathbf{Q}$, is A closed under $+$?

   (e) If $A = \mathbf{R} - \mathbf{Q}$, is A closed under $+$?

7. Let $S = \{q + r\sqrt{2} \mid q, r \in \mathbf{Q}\}$ with the usual addition and multiplication of real numbers. Complete the following to establish that S is a commutative ring.

   (a) Show that $+$ and $\cdot$ are commutative.

   (b) Show that $+$ and $\cdot$ are associative.

   (c) Show that $+$ distributes over $\cdot$.

   (d) Show that S contains an additive identity and a multiplicative identity.

   (e) Show that each element of S has an additive inverse.

8. Let S be the set of all 2×2 matrices of real numbers. Let $+$ be the usual matrix addition from linear algebra, and define a new "componentwise" multiplication $\star$ as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \star \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae & bf \\ cg & dh \end{bmatrix}$$

(a) Are $+$ and $\star$ commutative?

(b) Are $+$ and $\star$ associative?

(c) Do $+$ and $\star$ satisfy the distributive property?

(d) Is there an additive identity and a multiplicative identity?

(e) Are there additive inverses?

(f) Is S with $+$ and $\star$ a commutative ring?

9. **BONUS:** Find the units digit of 42^{4017}.