Math 234

Equivalence Relations Day 23

1. Let $A = \{10, 11, 12, 13, 14\}$. The relation

$$\mathbf{R} = \{(10,10), (10,14), (11,11), (11,13), (12,12), (13,11), (13,13), (14,10), (14,14)\}$$

is an equivalence relation. What are the equivalence classes of **R**?

$\{10, 14\}$ $\{12\}$

$\{11, 13\}$

2. Let **E** be a relation on the set **Z** of all integers defined by

$$m \; \mathbf{E} \; n \Leftrightarrow 4 \mid (m - n).$$

(a) Prove that this relation **E** is an equivalence relation by showing it is reflexive, symmetric and transitive.

reflexive: $\forall\, n \in \mathbb{Z}$, $4 \mid n-n = 0$, so $n E n$.

symmetric: if mEn, then $4 \mid m-n$, so $4 \mid n-m$, which means nEm.

transitive: Suppose mEn. So $m-n = 4k$ for some $k \in \mathbb{Z}$
Suppose nEp. So $n-p = 4r$ for some $r \in \mathbb{Z}$.
Then $m-p = (m-n) + (n-p) = m-p = 4(k+r)$, so mEp.

(b) Describe the equivalence class $[0]$ of **E**.

All multiples of 4. $4\mathbb{Z}$

$\{4k \mid k \in \mathbb{Z}\}$ $\{\dots, -8, -4, 0, 4, 8, \dots\}$

(c) Describe the equivalence class $[1]$ of **E**.

$\{x \in \mathbb{Z} \mid x = 4a+1 \text{ for some } a \in \mathbb{Z}\}$

$\{k \in \mathbb{Z} \mid k-1 \bmod 4 = 0\}$ $\{k \in \mathbb{Z} \mid k \equiv 1 \pmod 4\}$

(d) Describe the equivalence class $[2]$ of **E**.

$\{k \in \mathbb{Z} \mid k = 4a+2 \text{ for some } a \in \mathbb{Z}\}$

same

(e) Describe the equivalence class $[-31]$ of **E**.

$-31 = -32 + 1 = -8(4) + 1$ so $[-31] = [1]$

(f) Describe all the equivalence classes of **E**.

$\{\text{all multiples of } 4\}$

$\{k \in \mathbb{Z} \mid k = 4a+1 \text{ for some } a \in \mathbb{Z}\}$

$\{k \in \mathbb{Z} \mid k = 4a+2 \text{ for some } a \in \mathbb{Z}\}$

$\{k \in \mathbb{Z} \mid k = 4a+3 \text{ for some } a \in \mathbb{Z}\}$

3. Let $A = \mathbf{Z} \times \mathbf{Z}$. Define a relation $\mathbf{R}$ on A as follows: For all (a, b) and (c, d) in A,

$$(a, b)\ \mathbf{R}\ (c, d) \Leftrightarrow a + d = c + b.$$

(a) Is it true that $(1, 2)\ \mathbf{R}\ (3, 4)$? How about $(-1, 4)\ \mathbf{R}\ (0, 5)$?

Yes, since $1+4 = 2+3$

Yes, since $-1+5 = 4+0$

$(1,0) \not\mathbf{R} (2,0)$

$(2,4) \not\mathbf{R} (1,4)$

(b) Is $\mathbf{R}$ reflexive?

Yes, $(a,b) \mathbf{R} (a,b)$ since $a+b = a+b \ \forall\ a,b \in \mathbb{Z}$.

(c) Is $\mathbf{R}$ symmetric?

Yes: If $(a,b) \mathbf{R} (c,d)$, then $(c,d) \mathbf{R} (a,b)$.

$a+d = b+c$ $\quad$ $c+b = a+d$

(d) Is $\mathbf{R}$ transitive?

Yes: If $(a,b) \mathbf{R} (c,d)$ and $(c,d) \mathbf{R} (e,f)$, then $(a,b) \mathbf{R} (e,f)$.

$a+d = b+c$ $\quad$ $c+f = d+e$ $\quad$ $a+f = b+e$

$a-b = c-d = c-d = e-f$ so $a-b = c-f$

(e) Is $\mathbf{R}$ an equivalence relation?

Yes!

(f) List four elements of $[(1, 3)]$.

$(0,2)$, $(17,19)$, $(25,27)$, $(-1,1)$

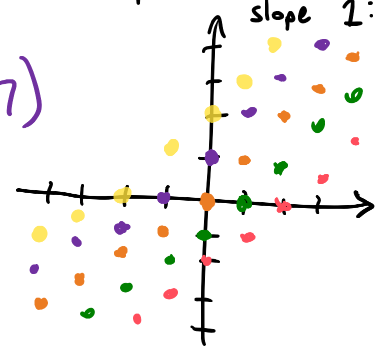
(g) List four elements of $[(-2, 6)]$.

$(-4,4)$, $(0,8)$, $(4,12)$, $(1729, 1737)$

(h) Describe all the equivalence classes of $\mathbf{R}$.

$[(a,b)] = \{(c,d) \in \mathbb{Z} \times \mathbb{Z} \mid d-c = b-a\}$

equivalence classes consist of points on lines of slope 1:



we didn't do this in class

4. Let X be a finite set. For all sets $U \in \mathscr{P}(X)$, let $N(U)$ denote the number of elements in U. Define a relation $\mathbf{R}$ on $\mathscr{P}(X)$ by $U\ \mathbf{R}\ V$ if and only if $N(U) = N(V)$.

Show that $\mathbf{R}$ is an equivalence relation. What are the equivalence classes of $\mathbf{R}$?

reflexive: $U \mathbf{R} U$ since $N(U) = N(U)$

symmetric: If $N(U) = N(V)$, then $N(V) = N(U)$ so $U \mathbf{R} V \Leftrightarrow V \mathbf{R} U$

transitive: If $N(U) = N(V)$ and $N(V) = N(W)$, then $N(U) = N(W)$.

so $U \mathbf{R} V$ and $V \mathbf{R} W \Rightarrow U \mathbf{R} W$.

5. Which of the following are partitions of the set $\mathbf{Z} \times \mathbf{Z}$ of ordered pairs of integers?

we didn't do this in class

(a) $\underbrace{\text{the set of pairs } (x, y) \text{ where } x \text{ or } y \text{ is odd}}_{A}$, $\underbrace{\text{the set of pairs } (x, y) \text{ where } x \text{ is even}}_{B}$, and $\underbrace{\text{the set of pairs } (x, y) \text{ where } y \text{ is even}}_{C}$

Not a partion: for example, (1,2) is in A and C

(b) the $\underbrace{\text{set of pairs } (x, y) \text{ where both } x \text{ and } y \text{ is odd}}_{A}$, $\underbrace{\text{the set of pairs } (x, y) \text{ where exactly one of } x \text{ and } y \text{ is odd}}_{B}$, $\underbrace{\text{and the set of pairs } (x, y) \text{ where both } x \text{ and } y \text{ are even}}_{C}$

Yes, this is a partition of $\mathbb{Z} \times \mathbb{Z}$.

(c) the $\underbrace{\text{set of pairs } (x, y) \text{ where } 3 \mid x \text{ and } 3 \mid y}_{A}$, the $\underbrace{\text{set of pairs } (x, y) \text{ where } 3 \mid x \text{ and } 3 \nmid y}_{B}$, $\underbrace{\text{the set of pairs } (x, y) \text{ where } 3 \nmid x \text{ and } 3 \mid y}_{C}$, the $\underbrace{\text{set of pairs } (x, y) \text{ where } 3 \nmid x \text{ and } 3 \nmid y}_{D}$

☞ the symbol $\nmid$ means "does not divide"

Yes, this is a partition of $\mathbb{Z} \times \mathbb{Z}$.

6. A partition P_1 is called a **refinement** of a partition P_2 if every set in P_1 is a subset of some set in P_2.

(a) Show that the partition formed from congruence classes modulo 6 is a refinement of the partition formed from congruence classes modulo 3. $[0]_6 \subseteq [0]_3$, $[1]_6 \subseteq [1]_3$, $[2]_6 \subseteq [2]_3$, $[3]_6 \subseteq [0]_3$, $[4]_6 \subseteq [1]_3$, $[5]_6 \subseteq [2]_3$

congruence classes mod 3:

$[0]_3 = \{\ldots, 0, 3, 6, 9, \ldots\}$, $[1]_3 = \{\ldots, 1, 4, 7, 10, \ldots\}$, $[2]_3 = \{\ldots, 2, 5, 8, 11, \ldots\}$

congruence classes mod 6: $[0]_6 = \{\ldots, 0, 6, 12, \ldots\}$, $[1]_6 = \{\ldots, 1, 7, 13, \ldots\}$, $[2]_6 = \{\ldots, 2, 8, 14, \ldots\}$

$[3]_6 = \{\ldots, 3, 9, 15, \ldots\}$, $[4]_6 = \{\ldots, 4, 10, 16, \ldots\}$, $[5]_6 = \{5, 11, 17, \ldots\}$

(b) If the partition formed from congruence classes modulo p is a refinement of the partition formed from congruence classes modulo q, what can you say about p and q?

q divides p.