Bifurcation Plane Lab

Math 230

due Thursday, March 17 at 4pm

In this lab you will investigate three families of first-order differential equations that each depend on two parameters, a and r. The goal in each case is to give a sketch of the "bifurcation plane" (or "parameter plane"). The bifurcation plane is a picture in the ar-plane of the regions (i.e., the values of a and r) for which there are different types of phase lines. The curves that separate these regions are the parameters where bifurcations occur (the "bifurcation curves").

Here are the three families:

Family 1: $\frac{dy}{dt} = r + ay - y^2$

Family 2: $\frac{dy}{dt} = ry + ay^2$

Family 3: $\frac{dy}{dt} = (y - r)(1 + ay + y^2)$

For each family, you should do the following:

1. Find all of the equilibrium solutions. These will, of course, depend on a and r. Identify how many equilibrium solutions exist for all pairs (a, r). Classify the equilibrium points.
2. Find the bifurcation curves. To do this, look for locations in the ar-plane where the number or type of equilibria changes. (For example, for what values of a and r do two equilibria come together?)
3. Sketch the bifurcation curves and the regions in the a, r-plane where you find different types of phase lines. In each different region, draw a representative picture of the phase line for any pair (a, r) in this region.
4. Finally, describe in a sentence or two the bifurcations that occur as you move from each region to an adjacent region.

An example of what is expected in this lab is given on the back of this page for the family $\frac{dy}{dt} = r + ay$.

It is recommended that you type your answers, though drawing the bifurcation planes by hand is completely acceptable. You may turn in your work either on paper or electronically on Moodle.

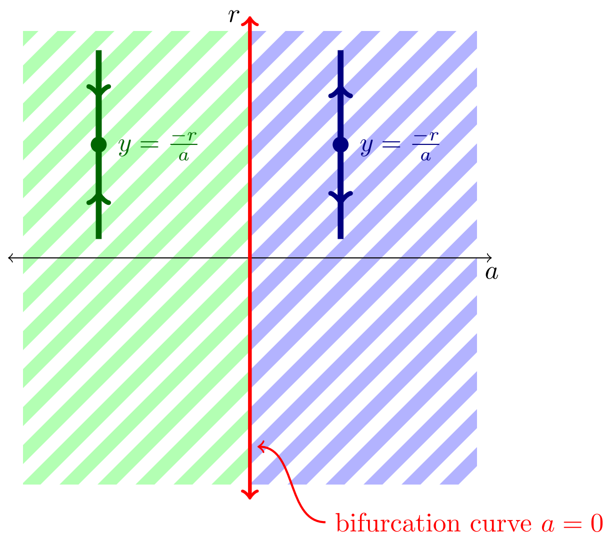
based on a lab by Robert L. Devaney: `http://math.bu.edu/people/bob/MA226/two-param-lab-2a.html`

Example

Here we give the bifurcation plane for $\frac{dy}{dt} = r + ay$. We first find all equilibrium solutions:

- If a is nonzero, then there is only one equilibrium solution, which is $y = -r/a$. This equilibrium point is a source if $a > 0$ and a sink if $a < 0$.
- If $a = 0$ and r is nonzero, then there is no equilibrium solution.
- If $a = r = 0$, then all points on the phase line are equilibria.

The bifurcation plane is illustrated here:



A bifurcation occurs as a passes through 0. There are three possible ways this can occur:

- If a passes through zero and $r > 0$, then the equilibrium point moves off to positive infinity when a approaches 0 from the negative side, and then reappears from negative infinity when a is positive. When $a = 0$, there is no equilibrium; all solutions are increasing functions.
- If a passes through zero and $r < 0$, then the equilibrium point moves off to negative infinity when a approaches 0 from the negative side, and then reappears from positive infinity when a is positive. When $a = 0$, there is no equilibrium; all solutions are decreasing functions.
- If the parameter pair (a, r) passes through $(0, 0)$, then there is a single equilibrium point when $a \neq 0$, but when $a = 0$ every point on the phase line becomes an equilibrium point.