POWER SERIES: Problem 2 from worksheet last time

$$\frac{d^2y}{dt^2} + t\frac{dy}{dt} + y = 1$$

Let $y(t) = \sum_{n=0}^{\infty} a_n t^n$. So $y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$ and $y''(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$

Plug in: $y'' + t y' + y = 1$

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + t\sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n = 1$$

let $k = n-2$

so $k+2 = n$, $k+1 = n-1$

$\sum_{n=1}^{\infty} n a_n t^n + a_0 + \sum_{n=1}^{\infty} a_n t^n$

$\sum (n a_n t^n + a_n t^n) = \sum (n+1) a_n t^n$

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} t^k + a_0 + \sum_{n=1}^{\infty} (n+1) a_n t^n = 1$$

when $k=0$

$$2a_2 + \sum_{k=1}^{\infty} (k+2)(k+1) a_{k+2} t^k + a_0 + \sum_{n=1}^{\infty} (n+1) a_n t^n = 1$$

Constants: $2a_2 + a_0 = 1$

let $k=n$

$n \geq 1$, t^n terms: $(n+2)(n+1) a_{n+2} + (n+1) a_n = 0$

$(n+2)(n+1) a_{n+2} = -a_n (n+1)$

$a_{n+2} = \frac{-a_n}{n+2}$

Unknowns: a_0, a_1

Then: $a_2 = \frac{1-a_0}{2}$, $\boxed{a_{n+2} = \frac{-a_n}{n+2}}$

Initial condition: $y(0) = 0$ and $y'(0) = 1$

$y(0)=0 \downarrow a_0 = 0$

$y'(0)=1 \downarrow a_1 = 1$

remember:

$y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$

$y'(0) = 1a_1 + 0$

$y'(0) = a_1$

$a_2 = \frac{1-a_0}{2} = \frac{1-0}{2} = \frac{1}{2}$

$a_3 = \frac{-a_1}{3} = \frac{-1}{3}$

$a_4 = \frac{-a_2}{4} = \frac{-\frac{1}{2}}{4} = \frac{-1}{8} \leftarrow 2(4)$

$a_5 = \frac{-a_3}{5} = \frac{-(\frac{-1}{3})}{5} = \frac{1}{15} \leftarrow 3(5)$

$a_6 = \frac{-a_4}{6} = \frac{-(\frac{-1}{8})}{6} = \frac{1}{48} \leftarrow 2(4)(6)$

$a_7 = \frac{-a_5}{7} = \frac{-1}{105} \leftarrow (3)(5)(7)$

What do you notice about $a_2, a_3, a_4, a_5, \ldots$

signs: $- - + + - - + +$ etc.

numerators: 1

denominators: double factorials

$n!! = (n)(n-2)(n-4)\cdots 2$ if n even

or $(n)(n-2)(n-4)\cdots(1)$ if n odd

CAUCHY-EULER EQUATION

1. $\boxed{t^2 \frac{d^2y}{dt^2}} - y = 0$ ← Note: $y=0$ is a solution!

(a) **Power series solution:** $y(t) = \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots$

Then: $y''(t) = 2a_2 + 6a_3 t + 12a_4 t^2 + \cdots$

Plug in: $t^2\left(2a_2 + 6a_3 t + 12a_4 t^2 + \cdots\right) - \left(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots\right) = 0$

$$2a_2 t^2 + 6a_3 t^3 + 12a_4 t^4 + \cdots - a_0 - a_1 t - a_2 t^2 - a_3 t^3 + \cdots = 0$$

constants: $-a_0 = 0$ so $a_0 = 0$

t: $-a_1 = 0$ so $a_1 = 0$

t^2: $2a_2 - a_2 = 0$ so $a_2 = 0$

t^3: $6a_3 - a_3 = 0$ so $a_3 = 0$

All the $a_n = 0$!

(c) **Try:** $y(t) = t^r$ **for some constant** r

$y'(t) = r t^{r-1}$

$y''(t) = r(r-1) t^{r-2}$

Plug in to $t^2 \frac{d^2y}{dt^2} - y = 0$:

$$t^2\left(r(r-1)t^{r-2}\right) - \left(t^r\right) = 0$$

$$r(r-1)\, t^r - t^r = 0$$

$$\left(r(r-1) - 1\right) t^r = 0$$

If $t \neq 0$, then $r(r-1) - 1 = 0$.

$$r^2 - r - 1 = 0$$

so $r = \dfrac{1 \pm \sqrt{1^2 - 4(-1)}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

We have: $\boxed{y_1(t) = c_1 t^{\frac{1+\sqrt{5}}{2}}}$ and $\boxed{y_2(t) = c_2 t^{\frac{1-\sqrt{5}}{2}}}$

Solutions to the diff. eq!

2. $t^2 \frac{d^2y}{dt^2} + 5t \frac{dy}{dt} - 5y = 0$

Try: $y(t) = t^r$:

$y'(t) = r t^{r-1}$

$y''(t) = r(r-1) t^{r-2}$

Plug in: $t^2\left(r(r-1)t^{r-2}\right) + 5t\left(r t^{r-1}\right) - 5t^r = 0$

$$r(r-1)\, t^r + 5r t^r - 5t^r = 0$$

$$t^r\,(r(r-1)+5r-5)=0$$

If $r \neq 0$: $r^2-r+5r-5=0$

$$r^2+4r-5=0$$

$$(r+5)(r-1)=0$$

so $r=1$ or -5

Solutions:

$y_1(t)=c_1 t$

$y_2(t)=c_2 t^{-5}$

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3. $t^2\frac{d^2y}{dt^2}-t\frac{dy}{dt}+5y=0$

Try: $y(t)=t^r$: $t^2\left(r(r-1)t^{r-2}\right)-t\left(rt^{r-1}\right)+5t^r=0$

$$t^r\left(r^2-r-r+5\right)=0$$

$$t^r\left(r^2-2r+5\right)=0$$

so $r=\dfrac{2\pm\sqrt{4-20}}{2}=\dfrac{2\pm 4i}{2}=1\pm 2i$

Solution: $y(t)=t^{1\pm 2i}$ ⟸ complex-valued

Find real solutions: $t^{1+2i}=t\cdot t^{2i}=t\,e^{\ln(t^{2i})}=t\,e^{2i\ln(t)}=t\left(\cos(2\ln t)+i\sin(2\ln t)\right)$

Linearly-independent, real-valued solutions:

$y_1(t)=c_1 t\cos(2\ln t)$, $y_2(t)=c_2 t\sin(2\ln t)$

4. $\frac{d^2y}{dt^2}+at\frac{dy}{dt}+by=0$

Try $y(t)=t^r$. Obtain the indicial equation: $r^2+(a-1)r+b=0$.

Three cases:

- Distinct roots r_1, r_2: $y=c_1t^{r_1}+c_2t^{r_2}$
- Repeated root r: $y=c_1t^r+c_2t^r\ln(t)$
- Complex roots $r=\alpha\pm\beta i$: $y=c_1t^\alpha\cos(\beta\ln t)+c_2t^\alpha\cos(\beta\ln t)$