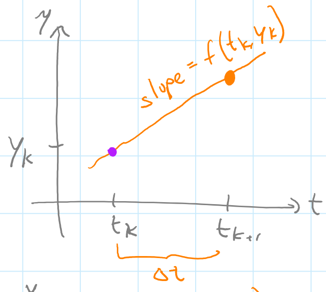
$$\frac{dy}{dt} = f(t,y), \quad y(0) = y_0$$

EULER'S METHOD:

$$y_{k+1} = y_k + f(t_k, y_k)\,\Delta t$$

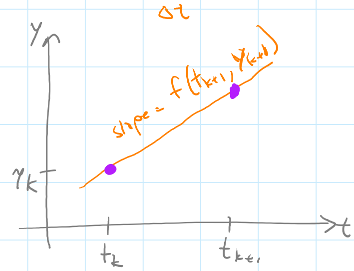


error: $O(\Delta t)$ "first order"

numerically unstable

BACKWARD EULER:

$$y_{k+1} = y_k + f(t_{k+1}, y_{k+1})\,\Delta t$$



error: $O(\Delta t)$

implicit method

example: $\frac{dy}{dt} = -10y, \quad y(0) = 1$

Euler's method is unstable for this diff. eq.

So: $f(t,y) = -10y$.

Then: $y_{k+1} = y_k + f(t_{k+1}, y_{k+1})\,\Delta t$

$$y_{k+1} = y_k - 10\,y_{k+1}\cdot\Delta t \quad \leftarrow \text{solve for } y_{k+1}$$

$$y_{k+1} + 10\,y_{k+1}\cdot\Delta t = y_k$$

$$y_{k+1}(1 + 10\Delta t) = y_k$$

$$\boxed{y_{k+1} = \frac{y_k}{1 + 10\Delta t}}$$

IMPROVED EULER METHOD

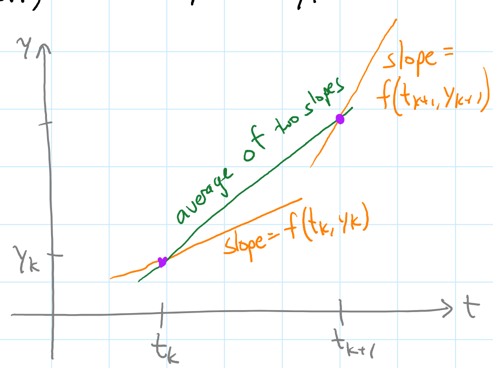
Use two slopes, at both t_k and t_{k+1}, to compute y_{k+1}.

Two step approach: For each k:

① First use Euler's method to compute a temporary value $\tilde{y}_{k+1}$

That is:

$$\tilde{y}_{k+1} = y_k + f(t_k, y_k)\,\Delta t$$



② Then, compute y_{k+1} using the average of the slopes at (t_k, y_k) and $(t_{k+1}, \tilde{y}_{k+1})$:

$$y_{k+1} = y_k + \underbrace{\frac{f(t_k, y_k) + f(t_{k+1}, \tilde{y}_{k+1})}{2}}_{\text{average of two slopes}}\Delta t$$

Second-Order Method

Error: $\underbrace{O(\Delta t^2)}_{\text{not more than a multiple of } \Delta t^2}$

Cut the step size in half, and the error is reduced by 4.

Example: $\frac{dy}{dt} = 1 - y, \quad y(0) = 0$ recall: exact solution: $y(t) = 1 - e^{-t}$

Example: $\frac{dy}{dt} = 2y - t, \quad y(0) = 1$ exact solution: $y(t) = \frac{1}{4}\left(3e^{2t} + 2t + 1\right)$

Approximate $y(1)$ using these 3 methods.

How do the errors depend on the step size?