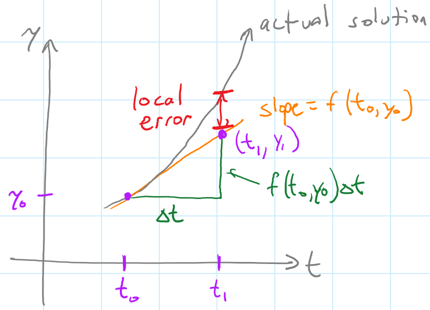
EULER'S METHOD

Recall: $\frac{dy}{dt} = f(t,y)$

$y(0) = y_0$

Approximate $y(a)$.



Approximation: $\boxed{y_n = y_{n-1} + f(t_{n-1}, y_{n-1})\,\Delta t}$

Local (truncation) error: difference between exact and approx. solutions at each step

Taylor series: $$y(t_1) = \underbrace{y(t_0) + y'(t_0)\,\Delta t}_{\text{linear function in } \Delta t} + \underbrace{\frac{y''(\xi)}{2}(\Delta t)^2}_{\text{ERROR TERM}}$$

Taylor series of order 1 with remainder term

approx

$\xi \in [t_0, t_0 + \Delta t]$

Euler's method: $$y_1 = y_0 + f(t_0, y_0)\,\Delta t$$

subtract: $$y(t_1) - y_1 = \cancel{y(t_0)} - \cancel{y_0} + \cancel{y'(t_0)\Delta t} - \cancel{f(t_0,y_0)\Delta t} + \frac{y''(\xi)}{2}(\Delta t)^2$$

$$y(t_1) - y_1 = \boxed{\frac{y''(\xi)}{2}}(\Delta t)^2$$

constant

error in Euler's method is equal to a constant times $(\Delta t)^2$

For small Δt, the error in Euler's method is $\underbrace{O(\Delta t^2)}$.

"big-O of Δt^2"

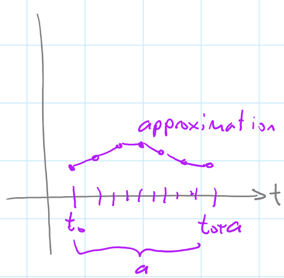
means: not greater than some multiple of Δt^2

Global error: Error in Euler's method from t_0 to $t_0 + a$.

Approximating $y(t_0+a)$ requires $\frac{a}{\Delta t}$ steps.

Total error is roughly $\frac{a}{\Delta t} \cdot O(\Delta t^2) = \underbrace{O(\Delta t)}$

not greater than some multiple of Δt



Euler's method is a "first order" approximation. ← This is not great. We can do better.

↳ $O(\Delta t)$

Example: $\frac{dy}{dt} = 1 - y$ $\xrightarrow{\text{solve}}$ $\int \frac{dy}{1-y} = \int dt$

$y(0) = 0$

Approximate $y(1)$.

$-\ln|1-y| = t + C$

$e^{\ln|1-y|} = e^{-t+C}$

$1 - y = e^{-t+C}$

$y = 1 - e^{-t+C}$

$y(0) = 0$ means

$0 = 1 - e^{0+C}$

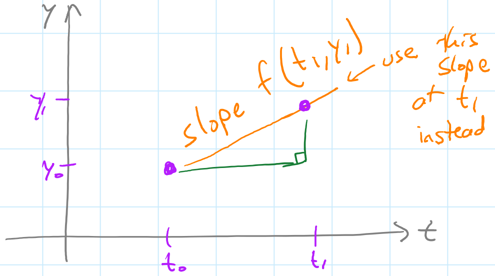
$e^C = 1$

so $C = 0$

exact solution: $\boxed{y(t) = 1 - e^{-t}}$

Problem: Euler's method is unstable for some equations, such as $\frac{dy}{dt} = -10y$.

IDEA: Use a different approximation scheme.



Then:

$$y_1 = y_0 + f(t_1, y_1)\,\Delta t$$

This results in the Backward Euler Method

If $f(t, y) = -10y$, then:

$$y_1 = y_0 - 10y_1 \cdot \Delta t$$

Need to solve for y_1.

To be continued...