HAMILTONIAN SYSTEMS

Recall: $\begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases}$ is Hamiltonian if there is a function $H(x,y)$ such that $f = \frac{\partial H}{\partial y}$ and $g = -\frac{\partial H}{\partial x}$.

QUESTION: How do we know if a system is Hamiltonian?

First, suppose there is a function H such that $f = \frac{\partial H}{\partial y}$, $g = -\frac{\partial H}{\partial x}$.

Then: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial y}\right) = \frac{\partial^2 H}{\partial x \partial y}$ ← These are equal

$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\partial H}{\partial x}\right) = -\frac{\partial^2 H}{\partial y \partial x}$ (assuming H has continuous second partial derivatives)

Thus: $\boxed{\text{If the system is Hamiltonian, then } \frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}.}$

Example: Is the system Hamiltonian?

$\begin{cases} \frac{dx}{dt} = x^2 + 2y = f \\ \frac{dy}{dt} = 2x - y^2 = g \end{cases}$

$\frac{\partial f}{\partial x} = 2x$ ←
$\frac{\partial g}{\partial y} = -2y$ ←

$\frac{\partial f}{\partial x} \neq -\frac{\partial g}{\partial y}$, so the system is <u>not</u> Hamiltonian.

Q: If $\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}$, then is the system Hamiltonian?

Can we find $H(x,y)$?

Example: $\frac{dx}{dt} = x^2 \cos y + 3y^2 = f$

$\frac{dy}{dt} = -2x \sin y + 3 = g$

$\frac{\partial f}{\partial x} = 2x \cos y$

$\frac{\partial g}{\partial y} = -2x \cos y$

$\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}$

Try to find $H(x,y)$ such that $f = \frac{\partial H}{\partial y}$ and $g = -\frac{\partial H}{\partial x}$.

Since $\frac{\partial H}{\partial y} = f$, integrate this with respect to y:

$$H(x,y) = \int f\, dy = \int (x^2 \cos y + 3y^2)\, dy$$

$$H(x,y) = x^2 \sin y + y^3 + C(x)$$

↑ "constant" of integration, constant with respect to y

Now use $g = -\frac{\partial H}{\partial x}$ to solve for $C(x)$:

$$-\frac{\partial H}{\partial x} = \underbrace{-\left(\cancel{2x\sin y} + C'(x)\right) = \cancel{-2x\sin y} + 3}\ = g(x,y)$$

$$-C'(x) = 3$$

$$C'(x) = -3$$

so $C(x) = \int -3\,dx = -3x + D$ ← constant

Thus: $H(x,y) = x^2 \sin y + y^3 - 3x + D$

For any D, this will have the property $f = \frac{\partial H}{\partial y}$, $g = -\frac{\partial H}{\partial x}$.

Choose $D=0$ for simplicity.

$$\boxed{H(x,y) = x^2 \sin y + y^3 - 3x}$$

Summary: If $\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}$, then the system is Hamiltonian. To find $H(x,y)$, first integrate $\frac{\partial H}{\partial y} = f(x,y)$ with respect to y, including a constant of integration $C(x)$ that depends on x. Then use $\frac{\partial H}{\partial x} = -g(x,y)$ to solve for $C(x)$.

Worksheet:

1. $\frac{dx}{dt} = y = f$

(a) $\frac{dy}{dx} = x^2 - a = g$ (a constant)

$\frac{\partial f}{\partial x} = 0$ and $\frac{\partial g}{\partial y} = 0$, so $\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}$

Hamiltonian function: $H(x,y) = \int f\,dy = \int y\,dy = \frac{1}{2}y^2 + C(x)$

Also: $\frac{\partial H}{\partial x} = -g(x,y) \implies$

$C'(x) = -(x^2 - a)$

so $C(x) = -\int (x^2 - a)\,dx = -\frac{1}{3}x^3 + ax$

Thus, $\boxed{H(x,y) = \frac{1}{2}y^2 - \frac{1}{3}x^3 + ax}$

(b) Equilibrium points: $y=0$

$$x^2 - a = 0 \implies x = \pm\sqrt{a}$$

equilibrium points: $(\sqrt{a},0)$ and $(-\sqrt{a},0)$ for $a \geq 0$

bifurcation at $a=0$

(c) Classify equilibrium points:

$$J(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2x & 0 \end{bmatrix}$$

At $(\sqrt{a},0)$: $J(\sqrt{a},0) = \begin{bmatrix} 0 & 1 \\ 2\sqrt{a} & 0 \end{bmatrix}$ Trace $=0$, Det $= -2\sqrt{a}$ SADDLE at $(\sqrt{a},0)$

At $(-\sqrt{a},0)$: $J(-\sqrt{a},0) = \begin{bmatrix} 0 & 1 \\ -2\sqrt{a} & 0 \end{bmatrix}$ Trace $=0$, Det $= 2\sqrt{a}$ CENTER at $(-\sqrt{a},0)$

Linearized System at $(-\sqrt{a},0)$: $\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -2\sqrt{a}\, x \end{cases}$

(d) As a increases through zero, an equilibrium point appears. The equilibrium point splits into a center and a saddle, which move apart along the horizontal axis as a increases.