COMPLEXIFICATION

Example: Find the general solution to $y'' + 3y' + 2y = e^{-2t}\cos(t)$

- Associated homogeneous eq: $y'' + 3y' + 2y = 0$

  characteristic polynomial: $\lambda^2 + 3\lambda + 2 = 0$

  $(\lambda+2)(\lambda+1) = 0$ so $\lambda = -1$ or -2

  So: $y_h(t) = k_1 e^{-t} + k_2 e^{-2t}$

- Nonhomogeneous eq: $y'' + 3y' + 2y = e^{-2t}\cos(t)$

  guess? we need to try $y_p(t) = Ae^{-2t}\cos(t) + Be^{-2t}\sin(t)$

  derivatives will be messy due to the product rule!

Observe: $y'' + 3y' + 2y = e^{-2t}\cos(t)$ is the real part of $y'' + 3y' + 2y = e^{(-2+i)t}$

Euler's formula

$e^{-2t}(\cos(t) + i\sin(t))$

Solve: $y'' + 3y' + 2y = e^{(-2+i)t}$

guess: $y_c(t) = Ae^{(-2+i)t}$

diff: $y_c'(t) = A(-2+i)e^{(-2+i)t}$

$y_c''(t) = A(-2+i)^2 e^{(-2+i)t}$

plug in:
$$y_c'' + 3y_c' + 2y_c = e^{(-2+i)t}$$
$$A(-2+i)^2\cancel{e^{(-2+i)t}} + 3A(-2+i)\cancel{e^{(-2+i)t}} + 2A\cancel{e^{(-2+i)t}} = \cancel{e^{(-2+i)t}}$$
$$A(-2+i)^2 + 3A(-2+i) + 2A = 1$$
$$A(4-4i-1) - 6A + 3Ai + 2A = 1$$
$$4A - 4Ai - A - 6A + 3Ai + 2A = 1$$
$$-A - Ai = 1$$
$$A(-1-i) = 1$$
$$A = \frac{1}{-1-i}\cdot\frac{-1+i}{-1+i} = \frac{-1+i}{1-i^2} = \frac{-1+i}{2}$$

So:
$$\boxed{y_c(t) = \frac{-1+i}{2}e^{(-2+i)t}}$$

Write $y_c(t) = \underline{y_{re}(t)} + i\,\underline{y_{im}(t)}$

real-valued functions

Since $y'' + 3y' + 2y = e^{-2t}\cos(t)$ is the real part of $y'' + 3y' + 2y = e^{(-2+i)t}$, the solution we want is the real part of $y_c(t)$.

Since $y''+3y'+2y = e^{-2t}\cos(t)$ is the real part of $y''+3y'+2y = e^{(-2+i)t}$,
the solution we want is the real part of $y_c(t)$.

Find the real part: $y_c(t) = \frac{-1+i}{2} e^{(-2+i)t} = \frac{-1+i}{2} e^{-2t}(\cos t + i\sin t)$

$$= \frac{1}{2}e^{-2t}(-1+i)(\cos t + i \sin t)$$

$$= \frac{1}{2}e^{-2t}(-\cos t - i\sin t + i\cos t - \sin t)$$

$$y_c(t) = \underbrace{\frac{1}{2}e^{-2t}(-\cos t - \sin t)}_{\substack{\text{real part} \\ y_{re}(t)}} + i\,\underbrace{\boxed{\frac{1}{2}e^{-2t}(-\sin t + \cos t)}}_{\substack{\text{imaginary part} \\ y_{im}(t)}}$$

- General solution to $y''+3y'+2y = e^{-2t}\cos t$:

$$\boxed{y(t) = k_1 e^{-t} + k_2 e^{-2t} + \frac{1}{2}e^{-2t}(-\cos t - \sin t)}$$

- General solution to $y''+3y'+2y = e^{-2t}\sin(t)$:

$$y(t) = k_1 e^{-t} + k_2 e^{-2t} + \frac{1}{2}e^{-2t}(-\sin t + \cos t)$$

$(v(t) = y'(t))$

What is the long-term behavior?

As $t \to \infty$, $(k_1 e^{-t} + k_2 e^{-2t}) \to 0$

So long-term behavior is periodic, approaches $\frac{1}{2}e^{-2t}(-\sin t + \cos t)$, and this also approaches zero.

Example: $y'' + 2y' + 2y = e^{-t}\sin t$

- associated homogeneous eq: $y'' + 2y' + 2y = 0$

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4-4(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2}$$

$$\lambda = -1 \pm i$$

So: $y_h(t) = k_1 e^{-t}\sin(t) + k_2 e^{-t}\cos(t)$

- nonhomogeneous eq: $y'' + 2y' + 2y = e^{-t}\sin t$ imaginary part of

complexification: $y'' + 2y' + 2y = e^{(-1+i)t}$

guess: $y_c(t) = A\,t\,e^{(-1+i)t}$

$y_c'(t) = A e^{(-1+i)t} + At(-1+i)e^{(-1+i)t} = A(1+t(-1+i))e^{(-1+i)t}$

guess: $y_c(t) = A\,t\,e^{(-1+i)t}$

$$y_c'(t) = Ae^{(-1+i)t} + At(-1+i)e^{(-1+i)t} = A\left(1+t(-1+i)\right)e^{(-1+i)t}$$

$$y_c''(t) = A(-1+i)e^{(-1+i)t} + A\left(1+t(-1+i)\right)(-1+i)e^{(-1+i)t}$$

$$= A\left((-1+i)\left(1+1+t(-1+i)\right)\right)e^{(-1+i)t}$$

$$= A(-1+i)(2-t+ti)e^{(-1+i)t}$$

plug in: $y_c'' + 2y_c' + 2y_c = e^{(-1+i)t}$

$$A(-1+i)(2-t+ti)\cancel{e^{(-1+i)t}} + 2A(1-t+ti)\cancel{e^{(-1+i)t}} + 2At\cancel{e^{(-1+i)t}} = \cancel{e^{(-1+i)t}}$$

$$A(-2\cancel{+t}-ti+2i-ti\cancel{-t}) + 2A(1-t+ti) + 2At = 1$$

$$A(\cancel{-2}\cancel{-2ti}+2i+\cancel{2}\cancel{-2t}\cancel{+2ti}\cancel{+2t}) = 1$$

$$2iA = 1$$

$$A = \frac{1}{2i}\cdot\frac{-i}{-i} = \frac{-i}{2}$$

So: $$\boxed{y_c(t) = \frac{-i}{2}t\,e^{(-1+i)t}}$$

or: $$y_c(t) = \frac{-i}{2}t\,e^{-t}(\cos t + i\sin t)$$

$$\boxed{y_c(t) = \frac{1}{2}te^{-t}\sin(t) - \frac{i}{2}te^{-t}\cos(t)} \longrightarrow$$ imaginary part is $-\frac{1}{2}te^{-t}\cos(t)$

Therefore, the general solution to $y''+2y'+2y = e^{-t}\sin t$ is:

$$\boxed{y(t) = k_1e^{-t}\sin(t) + k_2e^{-t}\cos(t) - \frac{1}{2}te^{-t}\cos(t)}$$