Ch. 4: Forced Oscillation

Second-order nonhomogeneous linear diff. eq.

(nonhomogeneous: different than in Ch. 3)

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t)$$

Strategy for solving:

1. Find the general solution to the associated homogeneous equation. (Ch. 3)
$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0 \xrightarrow[\text{solve}]{} k_1 y_1(t) + k_2 y_2(t) = y_h(t)$$

2. Find any particular solution $y_p(t)$ to the nonhomogeneous equation.

3. Add $y_h(t) + y_p(t)$ to obtain the general solution to the nonhomogeneous equation.

Example:

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = e^{-2t}$$

Associated homogeneous eq: $y'' + 4y' + 3y = 0$

characteristic polynomial: $\lambda^2 + 4\lambda + 3 = 0$

$(\lambda+3)(\lambda+1) = 0$ so $\lambda = -1$ or -3

general solution: $y_h(t) = k_1 e^{-t} + k_2 e^{-3t}$

Nonhomogeneous eq: $y'' + 4y' + 3y = e^{-2t}$

Try: $y_p(t) = Ae^{-2t}$

then: $y_p'(t) = -2Ae^{-2t}$

$y_p''(t) = 4Ae^{-2t}$

plug in:

$y_p'' + 4y_p' + 3y_p = e^{-2t}$

$4A\cancel{e^{-2t}} + 4(-2A\cancel{e^{-2t}}) + 3(A\cancel{e^{-2t}}) = \cancel{e^{-2t}}$

$4A - 8A + 3A = 1$

$-A = 1$

so $A = -1$

Thus, $y_p(t) = -e^{-2t}$

General solution to the nonhomogeneous equation:

$$y(t) = k_1 e^{-t} + k_2 e^{-3t} - e^{-2t}$$

WORKSHEET

eigenvalues: $\lambda = 0 \pm 2i$

Solution: $e^{0t}\sin(2t)$, $e^{0t}\cos(2t)$

1. $y'' + 4y = \cos(t)$

$y_h(t) = k_1 \cos(2t) + k_2 \sin(2t)$ gen. sol. to $y'' + 4y = 0$

Find $y_p(t)$: $y_p(t) = A\cos(t) + B\sin(t)$

$y_p'(t) = -A\sin(t) + B\cos(t)$

$y_p''(t) = -A\cos(t) - B\sin(t)$

Plug in: $y_p''(t) + 4y_p(t) = \cos(t)$

$-A\cos(t) - B\sin(t) + 4(A\cos(t) + B\sin(t)) = \cos(t)$

cos(t) coefficients: $-A + 4A = 1$, $3A = 1$, $A = \frac{1}{3}$

sin(t) Coeff: $-B + 4B = 0$, $3B = 0$, $B = 0$

So $y_p(t) = \frac{1}{3}\cos(t)$

General solution to the nonhomogeneous eq:

$y(t) = k_1\cos(2t) + k_2\sin(2t) + \frac{1}{3}\cos(t)$

2. $y'' + 2y' + y = e^{-t}$

$y'' + 2y' + y = 0 \longrightarrow$ repeated eigenvalue of $\lambda = -1$

so general solution is $y_h(t) = k_1 e^{-t} + k_2 t e^{-t}$

Find any solution to $y'' + 2y' + y = e^{-t}$

Try: $y_p(t) = At^2 e^{-t}$

$y_p'(t) = 2Ate^{-t} - At^2e^{-t} = (2At - At^2)e^{-t}$

$y_p''(t) = (2A - 2At)e^{-t} + (-2At + At^2)(+e^{-t}) = (At^2 - 4At + 2A)e^{-t}$

Plug in: $y_p'' + 2y_p' + y_p = e^{-t}$

$(At^2 - 4At + 2A)\cancel{e^{-t}} + 2(2At - At^2)\cancel{e^{-t}} + At^2\cancel{e^{-t}} = \cancel{e^{-t}}$

$At^2 - 4At + 2A + 4At - 2At^2 + At^2 = 1$

t^2 terms: $A - 2A + A = 0$
$0 = 0$ ✓

t terms: $-4A + 4A = 0$
$0 = 0$

constants:
$2A = 1$
$A = \frac{1}{2}$

So: $y_p(t) = \frac{1}{2}t^2 e^{-t}$

General solution to $y'' + 2y' + y = e^{-t}$:

$$y(t) = k_1 e^{-t} + k_2 t e^{-t} + \frac{1}{2}t^2 e^{-t}$$

3. $y'' + 6y' + 8y = 2t + e^t$

$y_h(t) = k_1 e^{-t} + k_2 e^{-4t}$

For y_p, guess $y_p(t) = At + B + Ce^t$, find $A = \frac{1}{4}$, $B = \frac{-3}{16}$, $C = \frac{1}{15}$

General solution: $y(t) = k_1 e^{-t} + k_2 e^{-4t} + \frac{1}{4}t - \frac{3}{16} + \frac{1}{15}e^t$

4. $y'' + 5y' + 6y = \sin(t)$

$y_h(t) = k_1 e^{-2t} + k_2 e^{-3t}$

For y_p, guess $y_p(t) = A\sin(t) + B\cos(t)$, find $A = \frac{1}{10}$, $B = \frac{-1}{10}$

General solution: $y(t) = k_1 e^{-2t} + k_2 e^{-3t} - \frac{1}{10}\cos t + \frac{1}{10}\sin t$