Repeated Eigenvalues:

A: 2×2 matrix with only 1 real eigenvalue (algebraic multiplicity 2)

Case 1: Only 1 linearly independent eigenvector

General solution: $\vec{Y}(t) = \vec{V}e^{\lambda t} + \vec{W}te^{\lambda t}$ $\quad\lambda$ is the eigenvalue

If $\vec{Y}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, then: $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \vec{V}\underbrace{e^{\lambda(0)}}_{1} + \underbrace{\cancel{\vec{W}(0)e^{\lambda(0)}}}_{0}$

so $\vec{V} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and $\vec{W} = (\mathbf{A} - \lambda\mathbf{I})\vec{V}$.

WORKSHEET

1. $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$ $\qquad \det(\mathbf{A}-\lambda\mathbf{I}) = \begin{bmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda) - (-1) = \lambda^2 - 4\lambda + 3 + 1$

$$= \lambda^2 - 4\lambda + 4 = (\lambda-2)^2 = 0$$

so $\lambda = 2$ is the only eigenvalue.

eigenvectors: $(\mathbf{A} - 2\mathbf{I})\vec{v}_1 = \vec{0}$

$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\vec{v}_1 = \vec{0}$ so $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or any multiple of this.

$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, $-x+y=0$

$$\boxed{\text{General solution: } \vec{Y}(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} e^{2t} + \vec{W}te^{2t}, \text{ where } \vec{W} = (\mathbf{A} - 2\mathbf{I})\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}}$$

2. (a) $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so $\vec{Y}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + \vec{0}$ $\qquad \vec{W} = (\mathbf{A} - 2\mathbf{I})\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{0}$

If $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is an eigenvector, then $\vec{W} = \vec{0}$.

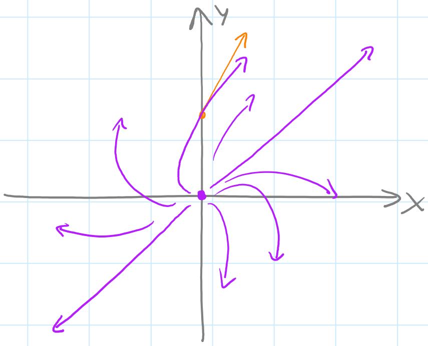
(b) $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ so $\vec{Y}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{2t}$ $\qquad \vec{W} = (\mathbf{A} - 2\mathbf{I})\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\vec{W} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

If $\vec{W} \neq 0$, then $\vec{W}$ is an eigenvector.

(c) $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ So $\vec{Y}(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} e^{2t} + \begin{bmatrix} -x_0 + y_0 \\ -x_0 + y_0 \end{bmatrix} te^{2t}$

3.



$\frac{d\vec{y}}{dt} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \vec{y}$

if $x=0$, $y=1$, then

$\frac{dx}{dt} = 1(0) + 1(1) = 1$

$\frac{dy}{dt} = -1(0) + 3(1) = 3$

Case 2: Two linearly independent eigenvectors

4. B is a 2×2 matrix with only 1 eigenvalue λ and two linearly independent eigenvectors

↳ Then every nonzero vector is an eigenvector for eigenvalue λ!

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

Why? Since B has two lin. indep eigenvectors, it is diagonalizable.

$P^{-1}BP = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \lambda I$, which implies $B = P(\lambda I)P^{-1} = \lambda I$

system: $\frac{d\vec{y}}{dt} = B\vec{y} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \vec{y}$

5. so: $\begin{cases} \frac{dx}{dt} = \lambda x(t) \\ \frac{dy}{dt} = \lambda y(t) \end{cases}$ solution: $\begin{cases} x(t) = k_1 e^{\lambda t} \\ y(t) = k_2 e^{\lambda t} \end{cases}$

ZERO EIGENVALUES (special case of real eigenvalues)

6. $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\det(C - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda + 1 - 1$

$= \lambda^2 - 2\lambda = \lambda(\lambda - 2) = 0$

so eigenvalues: $\lambda = 0$, $\lambda = 2$

eigenvectors: $\lambda_1 = 0$: $(C - 0I)\vec{v}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{v}_1 = 0$ so $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 2$: $(C - 2I)\vec{v}_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \vec{v}_2 = 0$ so $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

General Solution: $\vec{Y}(t) = k_1 \vec{v}_1 e^{\lambda_1 t} + k_2 \vec{v}_2 e^{\lambda_2 t}$

$$\vec{Y}(t) = k_1 \begin{bmatrix}1\\-1\end{bmatrix} \underbrace{e^{0t}}_{1} + k_2 \begin{bmatrix}1\\1\end{bmatrix} e^{2t}$$

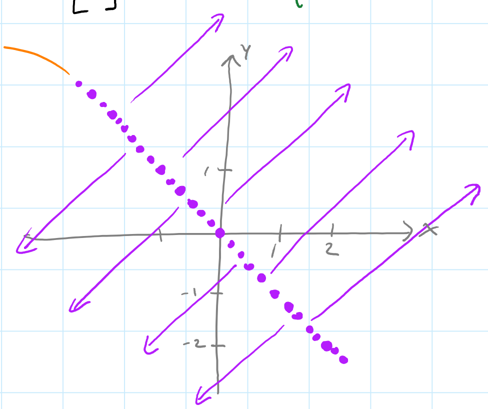
$$\boxed{\vec{Y}(t) = k_1 \begin{bmatrix}1\\-1\end{bmatrix} + k_2 \begin{bmatrix}1\\1\end{bmatrix} e^{2t}}$$

7. (a) $\vec{Y}(0) = \begin{bmatrix}2\\-2\end{bmatrix}$ then: $\begin{bmatrix}2\\-2\end{bmatrix} = \underbrace{k_1}_{k_1=2} \begin{bmatrix}1\\-1\end{bmatrix} + \underbrace{\cancel{k_2}}_{0} \begin{bmatrix}1\\1\end{bmatrix} e^{2(0)}$

initial condition is a multiple of the eigenvector for eigenvalue $\lambda = 0$.

Particular solution: $\vec{Y}(t) = 2 \begin{bmatrix}1\\-1\end{bmatrix}$ ← an equilibrium solution!

line of equilibrium solutions along $y = -x$



(b) $\vec{Y}(0) = \begin{bmatrix}3\\-1\end{bmatrix}$

Particular solution:

$$\vec{Y}(t) = \underbrace{2 \begin{bmatrix}1\\-1\end{bmatrix}}_{\text{constant}} + \underbrace{1 \begin{bmatrix}1\\1\end{bmatrix} e^{2t}}_{\substack{x = e^{2t} \\ y = e^{2t}}}$$

Linear Systems with Repeated or Zero Eigenvalues

Math 230

Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$.

1. Find the eigenvalues and eigenvectors of $\mathbf{A}$.

$$\det(A-\lambda I) = \begin{bmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$$

so the only eigenvalue is $\lambda = 2$

Eigenvector $\vec{V}$ satisfies $(A-2I)\vec{V} = \vec{0}$, so $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\vec{V} = \vec{0}$; thus $\vec{V} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
(or any multiple).

2. Find the solution with each of the following initial conditions:

(a) $\mathbf{Y}(0) = (1, 1)$
(b) $\mathbf{Y}(0) = (1, 2)$
(c) $\mathbf{Y}(0) = (x_0, y_0)$

(a) Since $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector,

$$\vec{Y}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$$

(b) Let $\vec{V}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and

$$\vec{V}_1 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The solution is:

$$\vec{Y}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{2t}$$

(c) Let $\vec{V}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and

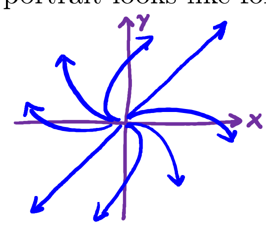
$$\vec{V}_1 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} y_0 - x_0 \\ y_0 - x_0 \end{bmatrix}.$$

The solution is:

$$\vec{Y}(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} e^{2t} + \begin{bmatrix} y_0 - x_0 \\ y_0 - x_0 \end{bmatrix} t e^{2t}.$$

Recall the Theorem from Section 3.5 in the text.

3. What do you think the phase portrait looks like for this system?



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4. Suppose a 2x2 matrix $\mathbf{B}$ has only one eigenvalue but *two* linearly independent eigenvectors. What can you say about matrix $\mathbf{B}$? (*Can you come up with any matrices with this property?*)

It must be that $B = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$.

Why? Since B has two linearly independent eigenvectors, it is diagonalizable.

Then $P^{-1}BP = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \lambda I$. This implies $B = P(\lambda I)P^{-1} = \lambda I$.

5. For any matrix $\mathbf{B}$ that you found above, what is the general solution to $\frac{d\mathbf{Y}}{dt} = \mathbf{BY}$?

The linear system is $\frac{d\vec{Y}}{dt} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\vec{Y}$, which has solution

$$x(t) = k_1 e^{\lambda t}, \quad y(t) = k_2 e^{\lambda t}.$$

(This system is completely decoupled.)

Consider the matrix $\mathbf{C} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

6. Find the eigenvalues and eigenvectors of $\mathbf{C}$.

$$\det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda = \lambda(\lambda-2)$$

so eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 2$

For $\lambda_1 = 0$, eigenvector is $\vec{V}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. For $\lambda_2 = 2$, eigenvector is $\vec{V}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(Or any multiple of these vectors.)

7. Find the solution with each of the following initial conditions:

(a) $\mathbf{Y}(0) = (2, -2)$

(b) $\mathbf{Y}(0) = (3, -1)$

(c) $\mathbf{Y}(0) = (x_0, y_0)$

General solution: $\vec{Y}(t) = k_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{0t} + k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$

$\vec{Y}(t) = k_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$

(a) Solution:

$Y(t) = \begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{0t} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

This is an equilibrium solution!

(b) $k_1 = 2$, $k_2 = 1$

Solution:

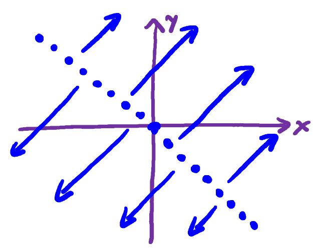
$\vec{Y}(t) = \begin{bmatrix} 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$

(c) $k_1 = \frac{x_0 - y_0}{2}$, $k_2 = \frac{x_0 + y_0}{2}$

Solution:

$\vec{Y}(t) = \frac{x_0 - y_0}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{x_0 + y_0}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$

8. What do you think the phase portrait looks like for this system?



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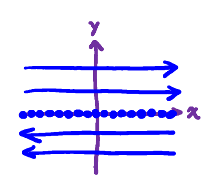
9. Suppose a 2x2 matrix $\mathbf{D}$ has only one eigenvalue, which is zero. What can you say about matrix $\mathbf{D}$? (*Can you come up with any matrices with this property?*)

Examples: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has only eigenvalue 0 and only one linearly independent eigenvector.

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the only matrix with only eigenvalue 0 and two linearly independent eigenvectors

10. For any matrix $\mathbf{D}$ that you found above, what is the general solution to $\frac{d\mathbf{Y}}{dt} = \mathbf{DY}$?

For $D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, the general solution is $\vec{Y}(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} y_0 \\ 0 \end{bmatrix} t$. phase portrait



For $D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, the general solution is $\vec{Y}(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$. ← Every point is an equilibrium point!