LINEAR SYSTEMS – COMPLEX EIGENVALUES

pure imaginary: $\lambda = \pm bi$

1. $A = \begin{bmatrix} -2 & 5 \\ -1 & 2 \end{bmatrix}$

eigenvalues: $\boxed{\lambda = \pm i}$ eigenvectors $\vec{v}_1, \vec{v}_2$

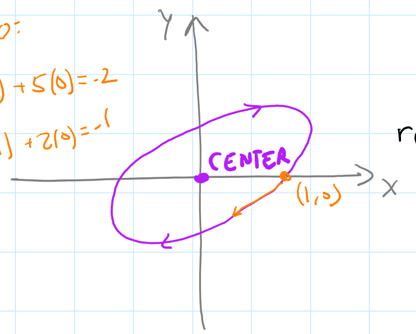
complex solution: $\vec{Y}(t) = k_1 \vec{v}_1 \underbrace{e^{it}}_{\cos t + i \sin t} + k_2 \vec{v}_2 \underbrace{e^{-it}}_{\cos(-t) + i\sin(-t)}$

$= \cos t - i \sin t$

At $x=1, y=0$:

$\frac{dx}{dt} = -2(1) + 5(0) = -2$

$\frac{dy}{dt} = -1(1) + 2(0) = -1$



real-valued solutions are linear combinations of $\cos(t)$ and $\sin(t)$

$$\begin{aligned} x(t) &= A\cos(t) + B\sin(t) \\ y(t) &= C\cos(t) + D\sin(t) \end{aligned}$$

periodic in t

period: 2π

2. $A = \begin{bmatrix} 1 & -2 \\ 2 & -2 \end{bmatrix}$

eigenvalues: $\lambda = \boxed{\frac{-1}{2}} \pm i \boxed{\frac{\sqrt{7}}{2}}$

real part

imaginary part

Complex eigenvalue, neg. real part

eigenvectors: $\vec{v}_1, \vec{v}_2$

solutions: $\vec{v}_1 e^{\left(-\frac{1}{2} \pm i \frac{\sqrt{7}}{2}\right)t} = \vec{v}_1 e^{-\frac{1}{2}t}\left(\cos\left(\frac{\sqrt{7}}{2}t\right) \pm i \sin\left(\frac{\sqrt{7}}{2}t\right)\right)$

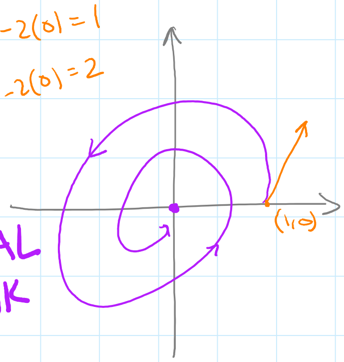
goes to zero as $t \to \infty$

$\underbrace{e^{-\frac{1}{2}t}}\ \underbrace{e^{\pm i \frac{\sqrt{7}}{2}t}}$

At $(1,0)$:

$\frac{dx}{dt} = 1(1) - 2(0) = 1$

$\frac{dy}{dt} = 2(1) - 2(0) = 2$



SPIRAL SINK

3. $A = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$

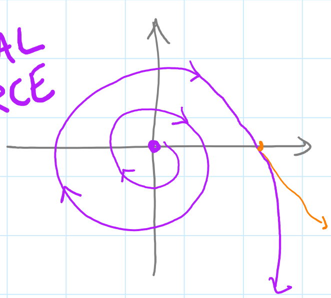
eigenvalues: $\lambda = \boxed{2} \pm i \boxed{\sqrt{7}}$

Complex eigenvalue, positive real part

Solutions: $\vec{v}_1 e^{(2 \pm i\sqrt{7})t} = \vec{v}_1 e^{2t}\left(\cos(\sqrt{7}t) \pm i \sin(\sqrt{7}t)\right)$

goes to ∞ as $t \to \infty$

SPIRAL SOURCE



At $(1,0)$:

$\frac{dx}{dt} = 1(1) + 4(0) = 1$

$\frac{dy}{dt} = -2(1) + 3(0) = -2$

→ REPEATED EIGENVALUES

Sometimes matrix $\mathbf{A}$ has a repeated eigenvalue.

That is: $\det(\mathbf{A} - \lambda \mathbf{I}) = (\lambda - c)^2$ for some $c \in \mathbb{R}$

$\lambda = c$

eigenvalue $\lambda = c$ has algebraic multiplicity 2

CASE 1: Only 1 direction of eigenvectors

example: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ eigenvalue: $\lambda = 1$, eigenvector: $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Solutions to $\frac{d\vec{Y}}{dt} = \mathbf{A}\vec{Y}$: $\vec{Y}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t$

Other solutions? Try: $\vec{Y}(t) = \vec{V} e^{\lambda t} + \vec{W} t e^{\lambda t}$

diff: $\frac{d\vec{Y}}{dt} = \lambda \vec{V} e^{\lambda t} + \vec{W}\left(e^{\lambda t} + \lambda t e^{\lambda t}\right)$

$\frac{d\vec{Y}}{dt} = (\lambda \vec{V} + \vec{W}) e^{\lambda t} + \lambda \vec{W} t e^{\lambda t}$

Plug in to $\frac{d\vec{Y}}{dt} = \mathbf{A}\vec{Y}$

$$(\lambda \vec{V} + \vec{W}) e^{\lambda t} + \lambda \vec{W} t e^{\lambda t} = \mathbf{A}\vec{V} e^{\lambda t} + \mathbf{A}\vec{W} t e^{\lambda t}$$

We need: $\lambda \vec{V} + \vec{W} = \mathbf{A}\vec{V}$ and $\lambda \vec{W} = \mathbf{A}\vec{W}$

$\vec{W} = (\mathbf{A} - \lambda \mathbf{I})\vec{V}$

- If $\vec{W} = \vec{0}$, then $\vec{V}$ is an eigenvector.
- If $\vec{W} \neq \vec{0}$, then solve for $\vec{V}$ ($\vec{V}$ is a generalized eigenvector.)

so $\vec{W}$ is an eigenvector (unless $\vec{W} = \vec{0}$)

Summary: If $\mathbf{A}$ has only one linearly independent eigenvector, then the general solution to $\frac{d\vec{Y}}{dt} = \mathbf{A}\vec{Y}$ is

$$\vec{Y}(t) = \vec{V} e^{\lambda t} + \vec{W} t e^{\lambda t}$$

QUESTION

If $\vec{Y}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, then how does this determine $\vec{V}$ and $\vec{W}$?