Note: $\underline{\frac{dy}{dt} = f(t,y)}$ is a differential equation

$f(t,y)$ is not a differential equation, it's a function of two variables

Solution to $\frac{dy}{dt} = f(t,y)$ is $y(t)$.

UNDETERMINED COEFFICIENT METHOD

$$\frac{dy}{dt} = a(t)y + b(t)$$

$b(t)$	typical "guess" $y_p(t)$
e^{at}	Ae^{at}
t^n	$A_0 + A_1t + A_2t^2 + \cdots + A_nt^n$
$\cos(at)$ or $\sin(at)$	$A\cos(at) + B\sin(at)$
$e^{at}\cos(bt)$ or $e^{at}\sin(bt)$	$Ae^{at}\cos(bt) + Be^{at}\sin(bt)$

Example: $\frac{dy}{dt} = 2y + e^t\cos(3t)$

$\frac{dy}{dt} = 2y$ has solution $y_h(t) = e^{2t}$

$$y_p(t) = Ae^t\cos(3t) + Be^t\sin(3t)$$

$$\frac{dy_p}{dt} = Ae^t\cos(3t) - 3Ae^t\sin(3t) + Be^t\sin(3t) + 3Be^t\cos(3t)$$

plug in to $\frac{dy}{dt} = 2y + e^t\cos(3t)$

$$A\cancel{e^t}\cos(3t) - 3A\cancel{e^t}\sin(3t) + B\cancel{e^t}\sin(3t) + 3B\cancel{e^t}\cos(3t) = 2A\cancel{e^t}\cos(3t) + 2B\cancel{e^t}\sin(3t) + \cancel{e^t}\cos(3t)$$

cos: $A + 3B = 2A + 1$

sin: $-3A + B = 2B$

$-3A = B$

$$A + 3(-3A) = 2A + 1$$
$$A - 9A = 2A + 1$$
$$-8A = 2A + 1$$
$$-10A = 1$$
$$A = \frac{-1}{10}$$

$$-3A = B$$
$$-3\left(\frac{-1}{10}\right) = B$$
$$\frac{3}{10} = B$$

$$\boxed{\begin{array}{c}\text{General solution to } \frac{dy}{dt} = 2y + e^t\cos(3t) \text{ is:}\\[1em] y(t) = \underset{k\cdot y_h(t)}{ke^{2t}} \underbrace{- \tfrac{1}{10}e^t\cos(3t) + \tfrac{3}{10}\sin(3t)}_{y_p(t)}\end{array}}$$

INTEGRATING FACTOR METHOD

$$\frac{dy}{dt} + g(t)y = b(t)$$

1. $\frac{dy}{dt} + \underbrace{\frac{2}{t}}_{g(t)} y = t$

Integrating factor:

$$\mu(t) = e^{\int g(t)\,dt} = e^{\int \frac{2}{t}\,dt} = e^{2\ln|t|}$$

$$= e^{\ln t^2} = t^2$$

$$t^2 \frac{dy}{dt} + t^2 \frac{2}{t} y = t^2 t$$

$$\underbrace{t^2 \frac{dy}{dt} + 2ty}_{} = t^3$$

$$\int \frac{d}{dt}(t^2 y)\,dt = \int t^3\,dt$$

$$t^2 y = \frac{1}{4}t^4 + C$$

So:

$$\boxed{y(t) = \frac{1}{4}t^2 + Ct^{-2}}$$

2. $\frac{dy}{dt} - 3t^2 y = e^{t^3}$

Integrating factor:

$$\mu(t) = e^{\int -3t^2\,dt} = e^{-t^3}$$

$$e^{-t^3}\frac{dy}{dt} - 3t^2 e^{-t^3} y = e^{t^3} e^{-t^3}$$

$$\int \frac{d}{dt}\left(e^{-t^3} y\right) dt = \int 1\,dt$$

$$e^{-t^3} y = t + C$$

$$\boxed{y(t) = (t+C)e^{t^3}}$$

3. $\frac{dy}{dt} - \frac{2}{t+2} y = (t+2)^3$

$$\mu(t) = e^{\int \frac{-2}{t+2}\,dt} = e^{-2\ln|t+2|} = (t+2)^{-2}$$

$$\frac{dy}{dt}(t+2)^{-2} - 2(t+2)^{-3} y = t+2$$

$$\int \frac{d}{dt}\left((t+2)^{-2} y\right) dt = \int (t+2)\,dt$$

$$(t+2)^{-2} y = \frac{1}{2}t^2 + 2t + C$$

so $\boxed{y(t) = (t+2)^2\left(\frac{1}{2}t^2 + 2t + C\right)}$