First-Order Linear Differential Equations

Math 230

Solve the linear differential equations by completing the following steps:

(a) Find the general solution to the associated homogeneous equation.

(b) Find a particular solution to the nonhomogeneous equation.

(c) Write down the general solution to the nonhomogeneous equation.

(d) Check your solution!

1. $\dfrac{dy}{dt} = 3y + e^{-t}$

The associated homogeneous equation $\frac{dy}{dt} = 3y$ has solution $y_h(t) = e^{3t}$.

For the particular solution, guess $y_p(t) = Ae^{-t}$. Substitute into the differential equation and solve to find $A = -\frac{1}{4}$.

Thus, the general solution is: $y(t) = ke^{3t} - \frac{1}{4}e^{-t}$

2. $\dfrac{dy}{dt} = -y + \sin(t)$

Associated homogeneous equation: $\frac{dy}{dt} = -y$ has solution $y_h(t) = e^{-t}$.

Particular solution: guess $y_p(t) = A \sin t + B \cos t$; substitute into the differential equation and solve to find $A = \frac{1}{2}$ and $B = -\frac{1}{2}$.

Thus, the general solution is: $y(t) = ke^{-t} + \frac{1}{2}\sin t - \frac{1}{2}\cos t$

3. $\dfrac{dy}{dt} = y + e^{t}$

Associated homogeneous equation: $\frac{dy}{dt} = y$ has solution $y_h(t) = e^{t}$.

Particular solution: we would like to guess e^t, but this is already y_h, so we instead choose $y_p(t) = Ate^t$. Substitute into the differential equation and solve to find $A = 1$.

Thus, the general solution is: $y(t) = ke^t + te^t$

Use an integrating factor to solve each linear differential equation:

1. $\dfrac{dy}{dt} + \dfrac{2y}{t} = t$

The integrating factor is $\mu(t) = e^{\int \frac{2}{t} dt} = e^{2 \ln t} = t^2$.

Thus, the differential equation is $\frac{d}{dt}\left(t^2 y\right) = t^3$.

Integrate to find that $y(t) = \frac{1}{4}t^2 + Ct^{-2}$.

2. $\dfrac{dy}{dt} = 3t^2 y + e^{t^3}$

First, rewrite the differential equation as $\dfrac{dy}{dt} - 3t^2 y = e^{t^3}$.

Then the integrating factor is $\mu(t) = e^{\int (-3t^2) dt} = e^{-t^3}$.

Thus, the differential equation is $\frac{d}{dt}\left(e^{-t^3} y\right) = 1$.

Integrate to find that $y(t) = (t + C)e^{t^3}$.

3. $\dfrac{dy}{dt} - \dfrac{2y}{t+2} = (t+2)^3$

The integrating factor is $\mu(t) = e^{\int \frac{-2}{t+2} dt} = e^{-2 \ln |t+2|} = (t+2)^{-2}$.

Thus, the differential equation is $\frac{d}{dt}\left((t+2)^{-2} y\right) = t + 2$.

Integrate to find that $y(t) = (t+2)^2 \left(\frac{1}{2}t^2 + 2t + C\right)$.