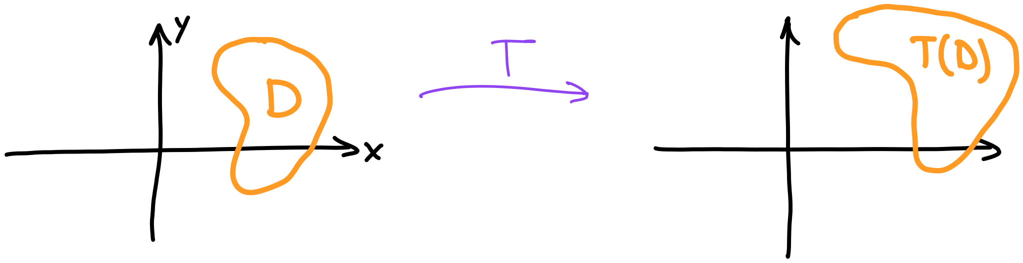
MATH 220

14 April 2025



Theorem 5.20: Let D be a region of finite area in $\mathbb{R}^2$. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(\vec{x}) = A\vec{x}$.

Let $T(D)$ denote the image of D under T.

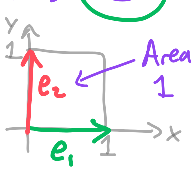
Then: $\text{area}(T(D)) = |\det(A)| \, \text{area}(D)$.

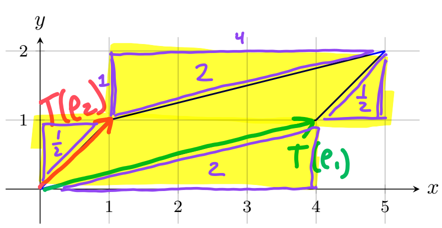
determinant is factor by which the area changes under the linear transformation.

6. Let $A = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$. Find $\det(A)$. The diagram shows the image of the unit square in first quadrant *after* being multiplied by A. What is the area of the parallelogram?

$\det(A) = 4(1) - (1)(1) = 3$

$A\vec{e}_1 = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$





Area of parallelogram:

$8 - (2 + 2 + \frac{1}{2} + \frac{1}{2})$

$= 8 - 5$

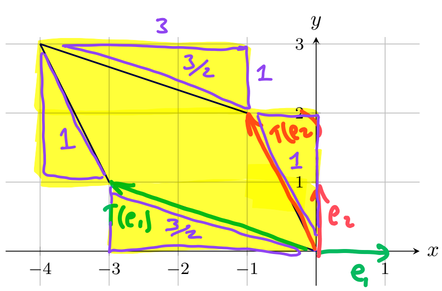
$= 3$

7. Let $B = \begin{bmatrix} -3 & -1 \\ 1 & 2 \end{bmatrix}$. Find $\det(B)$. The diagram shows the image of the unit square in first quadrant *after* being multiplied by B. What is the area of the parallelogram?

$\det(B) = -3(2) - (1)(-1)$

$= -6 + 1 = -5$

negative indicates a flip



Area of parallelogram:

$10 - (\frac{3}{2} + \frac{3}{2} + 1 + 1)$

$= 10 - (5) = 5$

8. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$. What is the connection between $\det(A)$ and the image of the unit square under T?

☞ If you aren't sure, experiment with more 2×2 matrices!

EIGENVECTORS

Let A be an $n \times n$ matrix. A nonzero vector $\vec{u}$ is an **eigenvector** of A if there exists a number λ such that

lambda

$$A\vec{u} = \lambda \vec{u}.$$

The number λ is called an **eigenvalue** of A.

Linear Algebra – Day 26

MATH 220

1. Cleo: We have seen this situation before!

Erez: Hmm...I think I would remember the word "eigenvector." We've never seen these.

Cleo: Think, Erez! We did not *call* them eigenvectors, but we have seen them already.

Erez: When? We've never worried about the equation $A\mathbf{x} = \lambda\mathbf{x}$.

Cleo (sighing loudly): But we did this in Chapter 3, for $\lambda = 1$.

Group chat: What "old" situation is Cleo trying to remind you of?

Steady-state vector from Markov chains: $A\vec{x} = 1\vec{x}$

$\vec{x}$ is eigenvector with eigenvalue 1

2. Let $A = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}$, $\quad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\quad \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\quad$ and $\quad \mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(a) **Split the work among your group members:** Compute $A\mathbf{u}$, $A\mathbf{v}$, and $A\mathbf{w}$.

$A\vec{u} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -1\vec{u}$ $\qquad$ $A\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq \lambda\vec{v}$ $\qquad$ $A\vec{w} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\vec{w}$

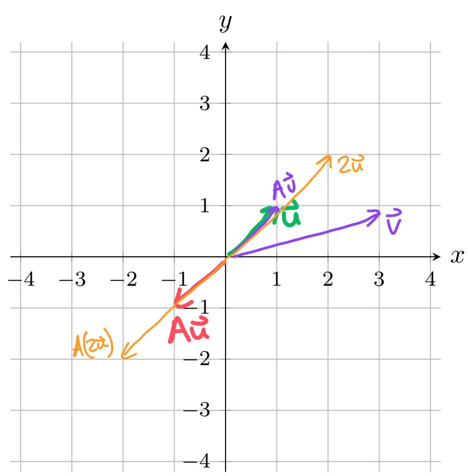
(b) **Group chat:** Do any of $\mathbf{u}, \mathbf{v}$, and/or $\mathbf{w}$ satisfy the definition of what it means to be an eigenvector of A? Why or why not?

☞ This should be easy to check. Use the definition of eigenvector.

$\vec{u}$ and $\vec{w}$ are eigenvectors

$\vec{v}$ is not an eigenvector

(c) Plot vectors $\mathbf{u}$, $\mathbf{v}$, $A\mathbf{u}$, and $A\mathbf{v}$ on the following coordinate plane.



(d) In part (b), you should have concluded that $\mathbf{u}$ is an eigenvector of A and $\mathbf{v}$ is not an eigenvector of A. Explain why your plots in part (c) also tell you this.

(e) Find an eigenvector of A *other than* $\mathbf{u}$ or $\mathbf{w}$.

☞ Don't guess and check, here. Remember that multiplication by A respects scalar multiplication.

3. Delphine, Milo, and Levi are now reflecting on all of their hard work on problem 2:

Delphine: OK, I see how this works. The vector $\mathbf{u}$ is an eigenvector because $A\mathbf{u} = (-1)\mathbf{u}$. That means that $\lambda = -1$ is an eigenvalue of A.

Milo: Indeed! Great work, Delphine. AND, the vector $\mathbf{w}$ is an eigenvector because $A\mathbf{w} = 0\mathbf{w}$. That means that $\lambda = 0$ is also an eigenvalue of A.

Delphine: That makes sense, because $A\mathbf{w}$ is definitely a multiple of $\mathbf{w}$ in that case.

Levi: Interesting...the number 0 can actually be an eigenvalue of a matrix. But wait, now I am confused. Why don't we want to let $\mathbf{0}$ be an eigenvector then? If we calculate $A\mathbf{0}$ we get $\mathbf{0}$, which is definitely a multiple of $\mathbf{0}$.

Group Chat: Why is it a good thing *not* to think of $\mathbf{0}$ as an eigenvector?

You should definitely read this conversation out loud at your table. Hearing written words spoken out loud can help with comprehension.

Hint: eigenvalue?

$\vec{0}$ is not special: $A\vec{0}$ is always $\vec{0}$

4. **Simon:** Hey! Did you know $\lambda = 3$ is an eigenvalue of $A = \begin{bmatrix} 4 & -2 & 5 \\ 1 & 1 & 5 \\ 1 & -2 & 8 \end{bmatrix}$?

Maura: How did you figure that out?

Simon: I have my ways. You'll have to trust me that I am correct.

Maura: I guess we should see if we can find an eigenvector that pairs with this eigenvalue.

Group chat: What does it *mean* to know that $\lambda = 3$ is an eigenvalue of A?

$A\vec{x} = 3\vec{x}$ has a nonzero solution (a vector $\vec{x}$ other than $\vec{0}$)

$\hookrightarrow (A-3I)\vec{x} = \vec{0}$

Group chat: What "equation" would you need to solve in order to find an eigenvector $\mathbf{x}$ that has eigenvalue $\lambda = 3$? Find at least one eigenvector $\mathbf{x}$!

Hint: null space?

null space of $(A-3I)\vec{x}$ has more than just the zero vector

class ended here

$$A - 3I = \begin{bmatrix} 1 & -2 & 5 \\ 1 & -2 & 5 \\ 1 & -2 & 5 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -2 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x_1 - 2x_2 + 5x_3 = 0$.

or $x_1 = 2x_2 - 5x_3$

For example, let $x_2 = 1$, $x_3 = 1$, and then $x_1 = -3$.

$\vec{x} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector

Simon: I didn't have time to check if $\lambda = 5$ is an eigenvalue of A or not. Is it?

Pretend $\lambda = 5$ is an eigenvalue and try to find some eigenvectors. What happens?

$$A - 5I = \begin{bmatrix} -1 & -2 & 5 \\ 1 & -4 & 5 \\ 1 & -2 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So $(A-5I)\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$, which means $\lambda = 5$ is <u>not</u> an eigenvalue of A.