MATH 220

11 April 2025

RECALL: DETERMINANT

Let A be a square $(n \times n)$ matrix. $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

Choose any row i.

Then: $\det(A) = (-1)^{i+1} a_{i1} \det(M_{i1}) + (-1)^{i+2} a_{i2} \det(M_{i2}) + \cdots + (-1)^{i+n} a_{in} \det(M_{in})$

remove row i
and column 1 from A

For 2×2 matrices, $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

Linear Algebra – Day 25

MATH 220

1. Find $\det(A)$ if $A = \begin{bmatrix} 2 & 5 & -1 \\ 4 & 0 & 3 \\ 1 & 0 & 6 \end{bmatrix}$.

row 2: $\det(A) = (-1)^{2+1}(4)\det\begin{bmatrix} 5 & -1 \\ 0 & 6 \end{bmatrix} + 0 + (-1)^{2+3}(3)\det\begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix}$

$= -4(30) - 3(-5) = -120 + 15 = \boxed{-105}$

col 2: $\det(A) = -5\det\begin{bmatrix} 4 & 3 \\ 1 & 6 \end{bmatrix} = -5(24-3) = -5(21) = \boxed{-105}$

2. Find $\det(A)$ if $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \\ 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \end{bmatrix}$

row 1: $\det(A) = (-1)^{1+1}(2)\det\begin{bmatrix} 3 & 1 & 8 \\ 0 & 0 & 5 \\ 7 & 2 & -5 \end{bmatrix}$

row 2: $= 2\left((-1)^{2+3}(5)\det\begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix}\right) = -10(6-7) = \boxed{10}$

3. (a) Find the determinant of each of the following matrices:

Pick a row which will make the calculation the easiest.

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \quad B = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}, \quad C = \begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix}$$

$\det(A) = ac$

$\det(B) = adf$

$\det(C) = aehj$

(b) **Theorem:** If A is a upper-triangular square matrix, then $\det(A)$ equals the product of the diagonal entries.

(c) What if A is a lower-triangular square matrix?

Same

(d) What if A is a diagonal square matrix?

$$\begin{bmatrix} a & & & & \\ & b & & 0 & \\ & & c & & \\ & 0 & & d & \\ & & & & e \end{bmatrix}$$

(e) What is the determinant of the identity matrix I_n?

1s on the diagonal...

$\det(I_n) = 1$

4. Find the determinant of each of the following matrices. You should be able to do this *without* the recursive definition (i.e., cofactor expansion). Explain.

$$A = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & 6 & 2 \\ 3 & 7 & 23 & -2 \\ 2 & 1 & 6 & 2 \\ \pi & e & \sqrt{17} & 0.02 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 3 & 0 & 3 \\ 2 & 3 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

zero row

equal rows

linearly dependent rows

determinant is zero in each case!

these matrices are not invertible!

5. Determine whether each statement is true or false, and explain your reasoning.

(a) $\det(A + B)$ equals $\det(A) + \det(B)$. No!

(b) $\det(A^T)$ equals $\det(A)$. Yes

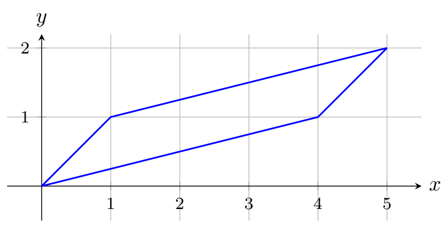
(c) $\det(AB)$ equals $\det(A)\det(B)$. Yes

(d) $\det(A^{-1})$ equals $\det(A)$ No ... but $\det(A^{-1}) = \dfrac{1}{\det(A)}$

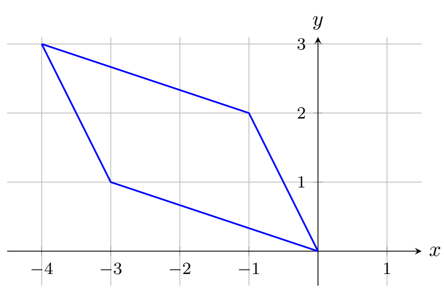
Gather evidence by calculating some small, specific examples. Remember, evidence is not proof that something is TRUE. But, evidence can show something is FALSE.

We will talk about these problems on Monday.

6. Let $A = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$. Find $\det(A)$. The diagram shows the image of the unit square in first quadrant *after* being multiplied by A. What is the area of the parallelogram?



7. Let $B = \begin{bmatrix} -3 & -1 \\ 1 & 2 \end{bmatrix}$. Find $\det(B)$. The diagram shows the image of the unit square in first quadrant *after* being multiplied by B. What is the area of the parallelogram?



8. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$. What is the connection between $\det(A)$ and the image of the unit square under T?

If you aren't sure, experiment with more 2×2 matrices!