MATH 220
9 April 2025

QUESTION: If A is an $n\times n$ matrix with $\underline{\text{rank}(A)=n}$, then what else do you know about A?

A is invertible

columns of A span $\mathbb{R}^n$, so do the rows

Columns are linearly independent, and so are the rows

Transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\vec{x}) = A\vec{x}$ is one-to-one and onto

$\text{nullity}(A) = 0$

RREF(A) has a pivot in every row and every column (RREF(A) $= I_n$ the identity matrix)

$A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$ in $\mathbb{R}^n$

DETERMINANTS

For any square matrix, there is a specific number, denoted $\underline{\det}(A)$, that tells you whether A is invertible.

determinant

If $\det(A) = 0$, then A^{-1} does not exist.
If $\det(A) \neq 0$, then A^{-1} exists.

For a 2x2 matrix: $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

How to compute determinants for matrices larger than 2x2?

Example 1: $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix}$

RECURSIVE APPROACH

compute 3x3 determinant in terms of 2x2 determinants

"Cofactor expansion"

Use Row 1:
$$\det(A) = (-1)^{1+1} 3 \det(M_{11}) + (-1)^{1+2} 1 \det(M_{12}) + (-1)^{1+3} 4 \det(M_{13})$$
$$= 3 \det\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} - 1 \det\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} + 4 \det\begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix}$$

Use Row 2:

row 2, col 1 — row 2, col 2 — row 2, col 3

$$\det(A) = (-1)^{2+1} 0 \det(M_{21}) + (-1)^{2+2} 1 \det(M_{22}) + (-1)^{2+3} 0 \det(M_{23})$$
$$= -0 \det\begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} + 1 \det\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} - 0 \det\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$
$$= 0 + 1(3\cdot 1 - 2\cdot 4) \quad 0$$

$\boxed{\det(A) = -5}$ not zero, so A is invertible

Example 2: $A = \begin{bmatrix} 9 & 2 & 3 & 2 \\ 2 & 0 & 2 & 5 \\ 0 & 3 & 0 & 0 \\ 3 & 1 & 0 & -1 \end{bmatrix}$

Use row 3:
$$\det(A) = \cancel{(-1)^{3+1}(0)\det(M_{31})} + (-1)^{3+2}(3)\det(M_{32}) + \cancel{(-1)^{3+3}(0)\det(M_{33})} + \cancel{(-1)^{3+4}(0)\det(M_{34})}$$
$$= -1(3)\det\begin{bmatrix} 9 & 3 & 2 \\ 2 & 2 & 5 \\ 3 & 0 & -1 \end{bmatrix}$$
$$= -3\left((-1)^{3+1}(3)\det(M_{31}) + 0 + (-1)^{3+3}(-1)\det(M_{33})\right)$$
$$= -3\left(3\det\begin{bmatrix} 3 & 2 \\ 2 & 5 \end{bmatrix} - \det\begin{bmatrix} 9 & 3 \\ 2 & 2 \end{bmatrix}\right)$$
$$= -3\left(3(3\cdot 5 - 2\cdot 2) - (9\cdot 2 - 2\cdot 3)\right) = -3\left(3(11) - (12)\right) = -3(21) = \boxed{-63}$$

Class ended here.

We didn't get to the worksheet that was posted on the course website.

We will do more with determinants on Friday.