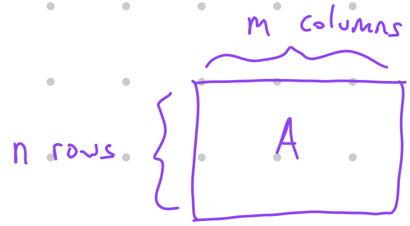
MATH 220
7 April 2025



Let A be an $n \times m$ matrix.
The span of the columns of A is the **Column Space**, denoted $\text{col}(A)$, and is a subspace of $\mathbb{R}^n$
The span of the rows of A is the **Row Space**, denoted $\text{row}(A)$, and is a subspace of $\mathbb{R}^m$.

- **RANK of a matrix is the common dimension of the row and column spaces.**

- **RANK-NULLITY THEOREM:** rank + nullity = number of columns

Linear Algebra – Day 23

MATH 220

1. For each matrix below, find a basis for the row space, a basis for the column space, and a basis for the null space.

Use Mathematica!

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 4 & 0 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 4 & 6 & 2 \\ 2 & 1 & 5 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{row}(A) = \text{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\} \qquad \text{basis: } \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$\text{col}(A) = \text{span}\left\{\begin{bmatrix}1\\2\\4\end{bmatrix}, \begin{bmatrix}2\\4\\0\end{bmatrix}\right\} \qquad \text{basis: } \begin{bmatrix}1\\2\\4\end{bmatrix}, \begin{bmatrix}2\\4\\0\end{bmatrix}$$

null space: $A\vec{x} = \vec{0}$

solutions: $x_1 + x_3 = 0 \Rightarrow x_1 = -x_3$

$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_3\begin{bmatrix}-1\\-1\\1\end{bmatrix}$$

$$\text{null}(A) = \text{span}\left(\begin{bmatrix}-1\\-1\\1\end{bmatrix}\right) \qquad \text{basis: } \begin{bmatrix}-1\\-1\\1\end{bmatrix}$$

dimensions

$\dim(\text{row}(A)) = 2$

$\dim(\text{col}(A)) = 2$

$\dim(\text{null}(A)) = 1$

$\text{rank}(A) = 2$

$\text{rank}(B) = 3$

B:

$$\text{col}(B) = \text{span}\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}4\\1\\3\end{bmatrix}, \begin{bmatrix}6\\5\\2\end{bmatrix}\right\} = \mathbb{R}^3 \qquad \text{row}(B) = \text{span}\left\{\begin{bmatrix}1\\4\\6\\2\end{bmatrix}, \begin{bmatrix}2\\1\\5\\0\end{bmatrix}, \begin{bmatrix}-1\\3\\2\\2\end{bmatrix}\right\}$$

$\dim(\text{col}(B)) = 3$ $\qquad$ $\dim(\text{row}(B)) = 3$

$$\text{null}(B) = \text{span}\left\{\begin{bmatrix}2\\-4\\0\\7\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}2/7\\-4/7\\0\\1\end{bmatrix}\right\}$$

$\dim(\text{null}(B)) = 1$

C:

$$\text{col}(C) = \text{span}\left\{\begin{bmatrix}1\\3\\-1\\2\end{bmatrix}, \begin{bmatrix}4\\-2\\0\\3\end{bmatrix}\right\}, \qquad \text{row}(C) = \text{span}\left\{\begin{bmatrix}1\\4\\5\\6\\9\end{bmatrix}, \begin{bmatrix}3\\-2\\1\\4\\-1\end{bmatrix}\right\}$$

$$\text{null}(C) = \text{span}\left\{\begin{bmatrix}-1\\-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\-2\\0\\0\\1\end{bmatrix}\right\}$$

$\dim(\text{col}(C)) = 2$

$\dim(\text{row}(C)) = 2$ $\qquad$ $\text{rank}(C) = 2$

$\dim(\text{null}(C)) = 3$

2. What do you observe about the dimensions of the spaces associated with each matrix above?

$$\underbrace{\dim(\text{row}(A))}_{\text{rank}} + \underbrace{\dim(\text{null}(A))}_{\text{nullity}} = \text{number of columns}$$

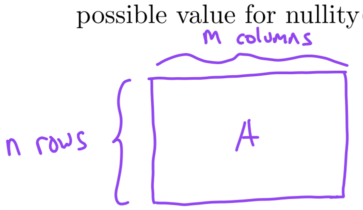
for all matrices A

$$\dim(\text{row}(A)) = \dim(\text{col}(A))$$

$\text{rank}(A) \le \min(n, m)$

minimum of n and m: whichever is smaller

3. If A is an $n \times m$ matrix, what is the largest possible value for rank(A)? What is the smallest possible value for nullity(A)?



row(A) is subspace of $\mathbb{R}^m$

col(A) is subspace of $\mathbb{R}^n$

if $n \ge m$, then nullity$(A) \ge 0$

if $m > n$, then nullity$(A) \ge m - n$

Class ended here.

4. For any matrix A, how does rank(A) relate to rank(A^T)?

These solutions are extra examples for practice.

$\text{rank}(A) = \text{rank}(A^T)$, since $\text{row}(A) = \text{col}(A^T)$ and $\text{col}(A) = \text{row}(A^T)$

5. (a) Give an example of a 3×3 matrix whose column space is a plane through the origin in $\mathbb{R}^3$.

One example is $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Then col(A) is the xy-plane in $\mathbb{R}^3$.

(b) What kind of geometric object is the null space of your matrix?

Null(A) is the z-axis, which is a line through the origin in $\mathbb{R}^3$.

(c) What kind of geometric object is the row space of your matrix?

Row(A) is the xy-plane in $\mathbb{R}^3$ (a plane through the origin).

6. Let $A = \begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix}$, where t is some number. What values of rank(A) are possible?

Row Reduce: $\begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & t \\ 0 & t-1 & 1-t \\ 0 & 1-t & 1-t^2 \end{bmatrix} \xrightarrow{\text{if } t \neq 1} \begin{bmatrix} 1 & 1 & t \\ 0 & 1 & -1 \\ 0 & 0 & 2-t-t^2 \end{bmatrix}$

$2 - t - t^2 = 0$ only when $t = 1$ or $t = -2$

If $t = 1$, then rank$(A) = 1$.

If $t = -2$, then rank$(A) = 2$.

For any other value of t, rank$(A) = 3$.

(Rank 0 is not possible since A has nonzero entries.)