MATH 220
26 March 2025

Null Space: subspace of vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$

Nullity: dimension of the nullspace.

Linear Algebra – Day 21

MATH 220

1. Find examples of 3×3 matrices A, B, C, and D such that:

$T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(\vec{x}) = A\vec{x}$

(a) $\text{nullity}(A) = 3$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) $\text{nullity}(B) = 2$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}$$

(c) $\text{nullity}(C) = 1$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(d) $\text{nullity}(D) = 0$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. If possible, find examples of $\overset{\text{rows}}{3} \times \overset{\text{cols}}{4}$ matrices A, B, C, D, and E such that:

$T: \mathbb{R}^4 \to \mathbb{R}^3$

$T(\vec{x}) = A\vec{x}$

(a) $\text{nullity}(A) = 4$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) $\text{nullity}(B) = 3$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) $\text{nullity}(C) = 2$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(d) $\text{nullity}(D) = 1$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(e) $\text{nullity}(E) = 0$

Not possible, since we cannot have 4 pivots in 3 rows.

3. Let A be an $n \times n$ matrix with column vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$.

List as statements as you can that are equivalent to the following.

The columns of A span $\mathbb{R}^n$.

Remember that "equivalent" means all true or all false.

The cols of A are lin. indep

T is onto.

T is one-to-one.

T is invertible.

$\text{nullity}(A) = 0$

$\ker(T) = \{\vec{0}\}$

The reduced form of A has a pivot in every row/column.

$A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^n$.

$A\vec{x} = \vec{0}$ has only the trivial solution.

Cols of A form a basis for $\mathbb{R}^n$.

Solution to $A\vec{x} = \vec{b}$ has no free variables.

4. Suppose all you know about A is that it is a $m \times n$ matrix, and let T be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$.

m rows, n columns $\qquad T: \mathbb{R}^n \to \mathbb{R}^m$

(a) What are the possible values for the dimension of the kernel of T?

$0, 1, 2, \ldots, n$

Note: if $m < n$ (more columns than rows), then the minimum dimension of the kernel is $n - m$.

(b) What are the possible values for the dimension of the range of T?

$0, 1, 2, \ldots, m$

(c) If you find out that the columns of A are linearly independent, how does that change your previous answers?

n pivots $\qquad \dim(\text{range}(A)) = n \qquad \text{nullity}(A) = 0$

num. rows is at least n

n cols

(d) If the columns of A are linearly independent, what else can you conclude about A and T?

T is one-to-one, but not onto unless $n = m$.

$A\vec{x} = \vec{b}$ has either zero or one solution for all $\vec{b}$ in $\mathbb{R}^m$.

The reduced form of A has a pivot in every column.