MATH 220

24 March 2025

RECALL: If S is a subspace, then a basis for S is a linearly independent set that spans S

All bases for S have exactly the same number of vectors, and this number is the dimension of S.

Example: $\dim(\mathbb{R}^n) = n$ standard basis $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$

$$\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} \quad \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}$$

$\dim(\mathbb{R}^1) = 1$

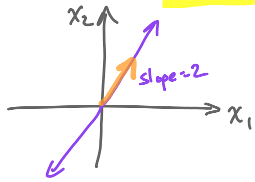
$\dim(\{\vec{0}\}) = 0$

There is no basis for $\{\vec{0}\}$.

Linear Algebra – Day 20

MATH 220

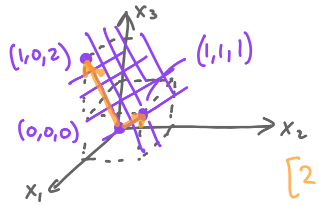
1. (a) The graph of $2x_1 - x_2 = 0$ is a line in $\mathbb{R}^2$. Sketch this line. Then find a basis for it.



$x_2 = 2x_1$

basis $\begin{bmatrix}1\\2\end{bmatrix}$

(b) The graph of $2x_1 - x_2 - x_3 = 0$ is a plane in $\mathbb{R}^3$. Try to sketch it. Then find a basis for it.



basis: $\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix} \right\}$

set brackets

$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \end{array}\right]$

(c) The graph of $2x_1 - x_2 - x_3 + x_4 = 0$ is a hyperplane (3 dimensions) in $\mathbb{R}^4$. Find a basis for it. 3 vectors?

$2(1) - 0 - 0 + (-2)$

$x_4 = -2x_1 + x_2 + x_3$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\\-2x_1+x_2+x_3\end{bmatrix} = x_1\begin{bmatrix}1\\0\\0\\-2\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\\1\end{bmatrix} + x_3\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

basis: $\left\{ \begin{bmatrix}1\\0\\0\\-2\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \right\}$

2. Let $A = \begin{bmatrix} 1 & 4 & 5 & 1 \\ 2 & 5 & 7 & 1 \\ 3 & 6 & 11 & 1 \end{bmatrix}$.

(a) Find null(A) by finding all solutions to $A\mathbf{x} = 0$.

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$x_1 = \frac{1}{3}x_4$

$x_2 = -\frac{1}{3}x_4$

$x_3 = 0$

$x_4 = x_4$

so nullspace consists of all vectors $x_4\begin{bmatrix}1/3\\-1/3\\0\\1\end{bmatrix}$ for $x_4 \in \mathbb{R}$

(b) Find a basis for null(A) by finding a linearly independent set of vectors that spans null(A). 1 vector?

$\left\{ \begin{bmatrix}1/3\\-1/3\\0\\1\end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix}1\\-1\\0\\3\end{bmatrix} \right\}$

(c) The **nullity** of A is the dimension of null(A). What is the nullity of A?

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(d) Find a basis for the range of A.

Span of columns of A: $\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}5\\7\\11\end{bmatrix} \right\}$

3. Find examples of 3×3 matrices A, B, C, and D such that:

We will return to this on Wednesday

(a) nullity(A) = 3

(b) nullity(B) = 2

(c) nullity(C) = 1

(d) nullity(D) = 0

We didn't get to this in class, but here is the solution for reference.

4. Consider the sets of vectors $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix} \right\}$ and $S' = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} \right\}$.

(a) How can you tell without doing any work that S is not a basis for $\mathbb{R}^3$?

S has only 2 vectors, but a basis for $\mathbb{R}^3$ must have 3 vectors.

(b) Is S linearly independent or dependent?

linearly independent, since the two vectors are not multiples of each other

(c) Construct a basis for $\mathbb{R}^3$ that includes the two vectors in S. How do you know you have a basis?

One basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ — row reduce $\begin{bmatrix} 1 & 1 & 0 \\ 2 & -4 & 0 \\ 3 & 5 & 1 \end{bmatrix}$ to confirm that the columns are linearly independent

(d) How can you tell without doing any work that S' is not a basis for $\mathbb{R}^3$?

S' has 4 vectors, but a basis for $\mathbb{R}^3$ must have exactly 3 vectors

(e) Does S' span all of $\mathbb{R}^3$?

Yes, since $\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & -4 & 8 & -1 \\ 3 & 5 & 1 & -3 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 0 & -101/22 \\ 0 & 1 & 0 & 39/22 \\ 0 & 0 & 1 & 21/11 \end{bmatrix}$, which has a pivot in each row

(f) Construct a basis for $\mathbb{R}^3$ that includes some of the vectors in S'. How do you know you have a basis?

Since the reduced matrix in part (e) has pivot entries in the first 3 columns, we can take the first 3 vectors in S': $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$

We will return to these on Wednesday

5. If possible, find examples of 3×4 matrices A, B, C, D, and E such that:

(a) $\text{nullity}(A) = 4$

(b) $\text{nullity}(B) = 3$

(c) $\text{nullity}(C) = 2$

(d) $\text{nullity}(D) = 1$

(e) $\text{nullity}(E) = 0$

6. Suppose all you know about A is that it is a $m \times n$ matrix, and let T be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$.

(a) What are the possible values for the dimension of the kernel of T?

(b) What are the possible values for the dimension of the range of T?

(c) If you find out that the columns of A are linearly independent, how does that change your previous answers?