MATH 220
21 March 2025

WARM-UP: Worksheet #1 and 2

FINDING A BASIS FOR A SUBSPACE S:

1. Figure out how to write S as a span
2. Choose as many linearly independent vectors as possible from the spanning set.

EXAMPLE: $S = \text{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\3\\3\end{bmatrix}, \begin{bmatrix}3\\4\\4\end{bmatrix}, \begin{bmatrix}3\\5\\5\end{bmatrix}\right)$. Find a basis for S.

ROW METHOD:

$$\begin{bmatrix}1&1&1\\2&3&3\\3&4&4\\3&5&5\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}1&0&0\\0&1&1\\0&0&0\\0&0&0\end{bmatrix} \qquad S = \text{span}\Bigg(\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}}_{\text{basis for } S}\Bigg)$$

COLUMN METHOD:

$$\begin{bmatrix}1&2&3&3\\1&3&4&5\\1&3&4&5\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}1&0&1&-1\\0&1&1&2\\0&0&0&0\end{bmatrix}$$

$$S = \text{span}\Bigg(\underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\3\\3\end{bmatrix}}_{\text{also a basis for } S}\Bigg)$$

Row operations change the span of the columns, but not the linear dependence or independence of the columns.

The dimension of S is 2, since any basis for S has 2 vectors.

Linear Algebra – Day 19

MATH 220

1. With your group, write down three random vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ (all in $\mathbb{R}^n$).

☞ Make it interesting, but simple.

(a) How are $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $\text{span}(\mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_1)$ related?

the spans are the same!

(b) How are $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $\text{span}(\mathbf{v}_1, 7\mathbf{v}_2, \mathbf{v}_3)$ related?

the same!

(c) How are $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 + 7\mathbf{v}_2)$ related?

the same!

2. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be the *rows* of a matrix A.

☞ THE ROWS. They are vectors, too.

(a) What happens to the span of the rows after the row operation $R_i \leftrightarrow R_j$?

☞ We did #1 for a reason.

span doesn't change!

(b) What happens to the span of the rows after the row operation $cR_i \to R_i$ ($c \neq 0$)?

☞ We did #1 for a reason.

span doesn't change!

(c) What happens to the span of the rows after the row operation $cR_i + R_j \to R_j$?

☞ We did #1 for a reason.

span doesn't change!

(d) Circle the one correct word:

Row operations *sometimes* *always* *never* change the span of the rows of a matrix.

3. For each statement, pick the correct symbol and discuss: $<$ $\leq$ $=$ $\geq$ $>$

(a) If you have m vectors that span $\mathbb{R}^3$, then $m \geq 3$.

(b) If you have m linearly independent vectors in $\mathbb{R}^3$, then $m \leq 3$.

(c) If you have m vectors that span $\mathbb{R}^n$, then $m \geq n$.

(d) If you have m linearly independent vectors in $\mathbb{R}^n$, then $m \leq n$.

(e) If you have m linearly independent that span $\mathbb{R}^n$, then $m = n$.

BASIS for a subspace S is a set of vectors that both spans S and is linearly independent.

4. How many vectors will there be in a basis for $\mathbb{R}^n$?

A basis for $\mathbb{R}^n$ must contain exactly n vectors

5. For this problem:

$$M = \begin{bmatrix} 1 & 2 & 4 & 7 \\ 2 & -1 & 3 & 5 \\ 4 & -7 & 1 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 & 3.4 \\ 0 & 1 & 1 & 1.8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -1 & -7 \\ 4 & 3 & 1 \\ 7 & 5 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Let $S = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -7 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix}\right)$ and $W = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 1 \\ 1 \end{bmatrix}\right)$.

- Using the "Row Method," a basis for W is: $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 3.4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1.8 \end{bmatrix}$

$\dim(W) = 2$

- Using the "Column Method," a basis for W is: $\begin{bmatrix} 1 \\ 2 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \\ 5 \end{bmatrix}$

- Using the "Row Method," a basis for S is: $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

$\dim(S) = 2$

- Using the "Column Method," a basis for S is: $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -7 \end{bmatrix}$

(b) **Cleo:** WOW! EVERYONE! Look at $\text{rref}(N) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. I can immediately see the number of vectors a basis of W will have!
Group chat: What is Cleo looking at?

The number of pivot entries.

(c) **Jonah:** AND, I can immediately see the number of vectors a basis of S will have!
Group chat: What is Jonah looking at?

The number of pivot entries.

(d) **Nadia: (looking quite excited):** I can ALSO immediately see the number of vectors a basis of null(N) will have.
Group chat: What is Nadia looking at?

The number of columns <u>without</u> pivot entries.