Linear Algebra – Day 18

MATH 220

1. Let $A = \begin{bmatrix} 1 & -1 & 1 & 4 & 4 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & -2 & -6 & 2 \end{bmatrix}$ and $\operatorname{rref}(A) = \begin{bmatrix} 1 & -1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) If $\mathbf{x}$ is a **solution** to $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x}$ is a vector in (choose one and explain):

$$\mathbb{R}^2 \qquad \mathbb{R}^3 \qquad \mathbb{R}^4 \qquad \mathbb{R}^5$$

(b) What are the free variables in the system $A\mathbf{x} = \mathbf{0}$?

(c) Write the solution set of $A\mathbf{x} = \mathbf{0}$ in *vector* form (in the space provided).

Remember, the "non-free" variables get solved for.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \\ \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \\ \\ \quad \end{bmatrix} + \begin{bmatrix} \quad \\ \\ \\ \\ \quad \end{bmatrix} + \begin{bmatrix} \quad \\ \\ \\ \\ \quad \end{bmatrix}$$

(d) Write the solution set to $A\mathbf{x} = \mathbf{0}$ *as a span of specific numerical vectors.*

Again, in the space provided.

$$\text{The solution set of } A\mathbf{x} = \mathbf{0} \text{ is span}\left(\begin{bmatrix} \quad \\ \\ \\ \\ \quad \end{bmatrix}, \begin{bmatrix} \quad \\ \\ \\ \\ \quad \end{bmatrix}, \begin{bmatrix} \quad \\ \\ \\ \\ \quad \end{bmatrix} \right).$$

(e) **Erez:** What we just did was not a coincidence! No matter what the matrix A is, we will always be able write the solutions to $A\mathbf{x} = \mathbf{0}$ as a span of some specific vectors!
Group chat: Discuss if/why Erez is right.

Erez is right.

Ava: As soon as I saw $\operatorname{rref}(A)$ at the start, I knew the solution set to $A\mathbf{x} = \mathbf{0}$ would be the span of 3 vectors.
Group chat: How can you tell ahead of time the *number* of vectors we would end up being included in the span that Ava is talking about?

Maura: AND, the vectors listed in the span are definitely linearly independent!
Group chat: Without doing arithmetic, how can you tell that the 3 vectors used in the span in part (c) are linearly independent?

(f) In this problem, you have found the *null space* of __________.

(g) In this problem, you have found the *kernel* of what?

2. Let J be the collection of all vectors in $\mathbb{R}^2$ of the form $\begin{bmatrix} x \\ x^2 \end{bmatrix}$.

(a) Find two vectors in $\mathbb{R}^2$ that are in J and two vectors in $\mathbb{R}^2$ that are NOT in J.

(b) **Erez:** Didn't we just learn that any collection we can write as a span is automatically a subspace?
Ava: Oh yeah...we should do that here! Every vector in J looks like

$$\begin{bmatrix} x \\ x^2 \end{bmatrix} = x \begin{bmatrix} 1 \\ x \end{bmatrix}.$$

Erez: Right, so $J = \text{span}\left(\begin{bmatrix} 1 \\ x \end{bmatrix}\right)$. By the theorem, J is a subspace of $\mathbb{R}^2$.

Group check Erez and Ava's work: Did they correctly conclude that J is a subspace of $\mathbb{R}^2$?

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(\mathbf{x}) = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \mathbf{x}$.

(a) What is the range of T?

(b) What is the kernel of T?

(c) Your answer to (b) tells you many things about T and A. List as many as you can!

4. Define the following subspaces:

$$S_1 = \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right) \qquad S_2 = \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}\right) \qquad S_3 = \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}\right)$$

$$S_4 = \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}\right) \qquad S_5 = \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}\right)$$

(a) All five of these collections are subspaces of $\mathbb{R}^3$. Why?
(b) What do S_1 and S_2 have in common? Is there anything different between them?
(c) What do S_2 and S_3 have in common? Is there anything different between them?
(d) What do S_3 and S_4 have in common? Is there anything different between them?
(e) What do S_4 and S_5 have in common? Is there anything different between them?