MATH 220
19 March 2025

## REVIEW:

A set $S$ of vectors in $\mathbb{R}^n$ is a <u>subspace</u> if:

1. $\vec{0}$ is in $S$
2. $S$ is closed under addition
   (if $\vec{u}$ and $\vec{v}$ are in $S$, then $\vec{u}+\vec{v}$ is in $S$)
3. $S$ is closed under scalar multiplication
   (if $\vec{u}$ is in $S$ and $c$ is any number, then $c \cdot \vec{u}$ is in $S$)

**THEOREM:** If $S = \text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k)$, then $S$ is a subspace.

**NULL SPACE** of a matrix $A$ is the subspace of solutions to $A\vec{x} = \vec{0}$.

Define linear transformation $T$ by $T(\vec{x}) = A\vec{x}$.

Then, the **KERNEL** of $T$ is the nullspace of $A$, the set of vectors $\vec{x}$ such that $T(\vec{x}) = \vec{0}$.

# Linear Algebra – Day 18

MATH 220

**1.** Let $A = \begin{bmatrix} 1 & -1 & 1 & 4 & 4 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & -2 & -6 & 2 \end{bmatrix}$ and $\text{rref}(A) = \begin{bmatrix} 1 & -1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

$x_1 = x_2 - x_4 - 5x_5$

$x_3 = -3x_4 + x_5$

(a) If $\mathbf{x}$ is a **solution** to $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x}$ is a vector in (choose one and explain):

$\mathbb{R}^2$ $\quad$ $\mathbb{R}^3$ $\quad$ $\mathbb{R}^4$ $\quad$ $\mathbb{R}^5$

(b) What are the free variables in the system $A\mathbf{x} = \mathbf{0}$?

3

(c) Write the solution set of $A\mathbf{x} = \mathbf{0}$ in *vector* form (in the space provided).

Remember, the "non-free" variables get solved for.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_2 - x_4 - 5x_5 \\ x_2 \\ -3x_4 + x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

(d) Write the solution set to $A\mathbf{x} = \mathbf{0}$ *as a span of specific numerical vectors.*

Again, in the space provided.

$\underbrace{\text{The solution set of } A\mathbf{x} = \mathbf{0}}_{\text{This is a subspace!}}$ is span $\left( \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$.

(e) **Erez:** What we just did was not a coincidence! No matter what the matrix $A$ is, we will always be able write the solutions to $A\mathbf{x} = \mathbf{0}$ as a span of some specific vectors!

**Group chat:** Discuss if/why Erez is right.

Erez is right.

Yes!

**Ava:** As soon as I saw $\text{rref}(A)$ at the start, I knew the solution set to $A\mathbf{x} = \mathbf{0}$ would be the span of 3 vectors.

**Group chat:** How can you tell ahead of time the *number* of vectors we would end up being included in the span that Ava is talking about?

number of free variables

**Maura:** AND, the vectors listed in the span are definitely linearly independent!

**Group chat:** Without doing arithmetic, how can you tell that the 3 vectors used in the span in part (c) are linearly independent?

Focus on rows with only one nonzero entry!

Is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ in nullspace of A?

(f) In this problem, you have found the *null space* of $\underline{\quad A \quad}$.

(g) In this problem, you have found the *kernel* of what?

The linear transformation $T(\vec{x}) = A\vec{x}$.

$$\begin{bmatrix} 1 & -1 & 1 & 4 & 4 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & -2 & -6 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

not $\vec{0}$

So $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is not in nullspace of A

We skipped this in class

**2.** Let $J$ be the collection of all vectors in $\mathbb{R}^2$ of the form $\begin{bmatrix} x \\ x^2 \end{bmatrix}$. not linear!

(a) Find two vectors in $\mathbb{R}^2$ that are in $J$ and two vectors in $\mathbb{R}^2$ that are NOT in $J$.

For example: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 4 \end{bmatrix}$ are in $J$, while $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 5 \end{bmatrix}$ are not in $J$

(b) **Erez:** Didn't we just learn that any collection we can write as a span is automatically a subspace?

**Ava:** Oh yeah...we should do that here! Every vector in $J$ looks like

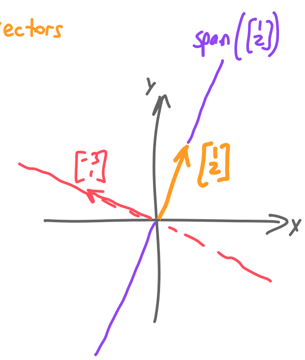
$$\begin{bmatrix} x \\ x^2 \end{bmatrix} = x \begin{bmatrix} 1 \\ x \end{bmatrix}.$$

**Erez:** Right, so $J = \text{span}\left(\begin{bmatrix} 1 \\ x \end{bmatrix}\right)$. By the theorem, $J$ is a subspace of $\mathbb{R}^2$.

**Group check Erez and Ava's work:** Did they correctly conclude that $J$ is a subspace of $\mathbb{R}^2$?

No, $x$ can't be both a value in the vector and a coefficient of the linear combination. When writing a span, we want to use vectors with specific numbers, not $x$.

**3.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(\mathbf{x}) = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \mathbf{x}$.




(a) What is the range of $T$?

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Range: $\text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$

all multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(b) What is the kernel of $T$?

Solutions to $A\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 6 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Solution: $x_1 = -3x_2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Kernel of $T$ is $\text{span}\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right)$

(c) Your answer to (b) tells you many things about $T$ and $A$. List as many as you can!

Since $\ker(T) \neq \{\vec{0}\}$, the Unifying Theorem tells us:

- The columns of $A$ don't span $\mathbb{R}^2$
- The columns of $A$ are not linearly independent
- $A\vec{x} = \vec{b}$ does not have a unique solution for all $\vec{b}$ in $\mathbb{R}^2$
- $T$ is not onto
- $T$ is not one-to-one
- $A$ is not invertible

We did not do this in class

**4.** Define the following subspaces:

$$S_1 = \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right) \quad S_2 = \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}\right) \quad S_3 = \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}\right)$$

$$S_4 = \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}\right) \quad S_5 = \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}\right)$$

(a) All five of these collections are subspaces of $\mathbb{R}^3$. Why? Yes, they are spans

(b) What do $S_1$ and $S_2$ have in common? Is there anything different between them?

$S_1$ is just multiples of a single vector, while $S_2$ is a plane.

(c) What do $S_2$ and $S_3$ have in common? Is there anything different between them?

$S_2$ and $S_3$ are the same span!

(d) What do $S_3$ and $S_4$ have in common? Is there anything different between them?

$S_2$ is a plane, but $S_3$ is all of $\mathbb{R}^3$

(e) What do $S_4$ and $S_5$ have in common? Is there anything different between them?

$S_4$ and $S_5$ are each all of $\mathbb{R}^3$