MATH 220
17 March 2025

DEFINITION OF A SUBSPACE:

Let S be a collection of vectors in $\mathbb{R}^n$.

Then S is a subspace if:

1. $\vec{0}$ is in S
2. S is closed under addition
   (If $\vec{u}$ and $\vec{v}$ are in S, then $\vec{u}+\vec{v}$ is in S.)
3. S is closed under scalar multiplication
   (If $\vec{u}$ is in S and c is any number, then $c \cdot \vec{u}$ is in S.)

Linear Algebra – Day 17

MATH 220

1. Let H be the collection of all vectors of the form $\begin{bmatrix} s \\ t \\ 0 \end{bmatrix}$ where s and t are real numbers.

☞ "Of the form" means "what the vectors look like."

(a) Just by looking at a vector, how can you tell whether or not the vector is in H?

Any vector in $\mathbb{R}^3$ whose 3rd entry is zero is in H.

example: $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ is in H, but $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is not in H

(b) Is the vector $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ in H ?

Yes, since its 3rd entry is 0.

> **Question #1:** If $\mathbf{u}$ is *any* vector in H and $\mathbf{v}$ is *any* vector in H, then will it always be true that $\mathbf{u} + \mathbf{v}$ is also a vector in H?
>
> **Question #2:** If $\mathbf{u}$ is *any* vector in H and c is *any* scalar, then it will always be true that $c \cdot \mathbf{u}$ is also a vector in H?

Is H closed under addition?

Is H closed under scalar multiplication?

(c) **Sundar:** The answer to Question #1 is "yes."

Marissa: Correct! To prove that, you just need to explain why $\begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ 0 \end{bmatrix}$ is in H.

Group chat: What is Marissa talking about? Is it true that $\begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ 0 \end{bmatrix}$ is in H? Why?

$$\begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ 0 \end{bmatrix}$$

Since 3rd entry is zero, this is in H

Now, we've shown that H is a subspace!

(d) **Group chat:** The answer to Question #2 is also "yes." What would you have to do in order to *show* that?

$$c \begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix} = \begin{bmatrix} c\,a_1 \\ c\,b_1 \\ 0 \end{bmatrix}$$

Since 3rd entry is zero, this is in H.

(e) **Leo:** I just noticed that H equals span$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$.

Group chat: Is Leo correct?

Yes. Any vector in H is also in span.
Any vector in this span is also in H.

$$s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} s \\ t \\ 0 \end{bmatrix}$$

Therefore, $H = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$.

(f) **Sundar:** Wow! H equals $\mathbb{R}^2$.

Nadia (shaking head sadly): Oh, Sundar...

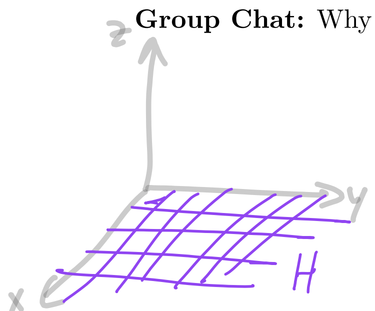
Group Chat: Why is Nadia sad? Or, is Sundar correct?

Every span is a subspace! (Theorem 4.2 in text)

Vectors in H have 3 components, so they are not in $\mathbb{R}^2$.

$\text{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$

H does not equal $\mathbb{R}^2$.



But, H is a 2-dimensional subspace of $\mathbb{R}^3$.

2. Let S be the collection of vectors in $\mathbb{R}^3$ of the form $\begin{bmatrix} x \\ x+1 \\ y \end{bmatrix}$.

Marissa: S is *not* a subspace of $\mathbb{R}^3$.

Group chat: Is Marissa correct? Investigate all three requirements for a subspace to see which one(s) go wrong.

- Is the zero vector, $\mathbf{0}$, in S?
- Can you find two vectors that are both in S, but their sum is NOT in S?
- Can you find a vector in S but some scalar multiple of that vector is NOT in S?

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not of the form $\begin{bmatrix} x \\ x+1 \\ y \end{bmatrix}$ since there is no choice of x, y that makes $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ x+1 \\ y \end{bmatrix}$. Thus, S is not a subspace

$\underset{\text{in } S}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}} + \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \underset{\text{NOT in } S}{\begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}}$ Thus, S is not closed under addition, so S is not a subspace.

3. Let K be the collection of vectors in $\mathbb{R}^3$ of the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where $x = 4z$ and $y = -3z$. Write K as a span of a single vector $\mathbf{d}$. Why does that show that K *is* a subspace of $\mathbb{R}^3$?

☞ Here, the "form" is not already "inside of" the vector. YOU have to put it there yourself.

K contains vectors $\begin{bmatrix} 4z \\ -3z \\ z \end{bmatrix}$ for any number z. $\begin{bmatrix} 4z \\ -3z \\ 1 \end{bmatrix} = z \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$

Thus, $K = \text{span}\left(\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}\right)$, and so K is a subspace.

4. **Group chat:** Is the set that *only contains the zero vector* (i.e., $\{\mathbf{0}\}$) a subspace of $\mathbb{R}^n$?

Yes: $\{\vec{0}\}$ satisfies the 3 properties of a subspace

Also, $\{\vec{0}\} = \text{span}(\vec{0})$.

5. **Group chat:** Is $\mathbb{R}^n$ a subspace of $\mathbb{R}^n$?

Yes, $\mathbb{R}^n$ satisfies the 3 properties of a subspace

6. **Spicy!** Does the collection of vectors of the form $\begin{bmatrix} a+b+2 \\ a-b-2 \\ 0 \end{bmatrix}$ a subspace of $\mathbb{R}^3$? Explain.

☞ Is this collection the span of some vectors?

See next page.

Let S be the collection of all vectors $\begin{bmatrix} a+b+2 \\ a-b-2 \\ 0 \end{bmatrix}$ for any numbers a and b.

First, note that each vector in S has the form $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ for some numbers x and y. So S is contained in $\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$.

Second, let x and y be any numbers We will show that $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ is in S.

To do this, we must solve $a+b+2=x$ and $a-b-2=y$.

This is equivalent to the system $\begin{cases} a+b=x-2 \\ a-b=y+2 \end{cases}$

which corresponds to the augmented matrix $\left[\begin{array}{cc|c} a & b & x-2 \\ a & -b & y+2 \end{array}\right]$

Solve to find $a=\frac{x+y}{2}$ and $b=\frac{x-y}{2}-2$, which implies that $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ is in S.

Since $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ is in S for all x and y, $\text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$ is contained in S.

Since each of S and $\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$ contains the other, they must be equal.

So $S=\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$, and therefore S is a subspace