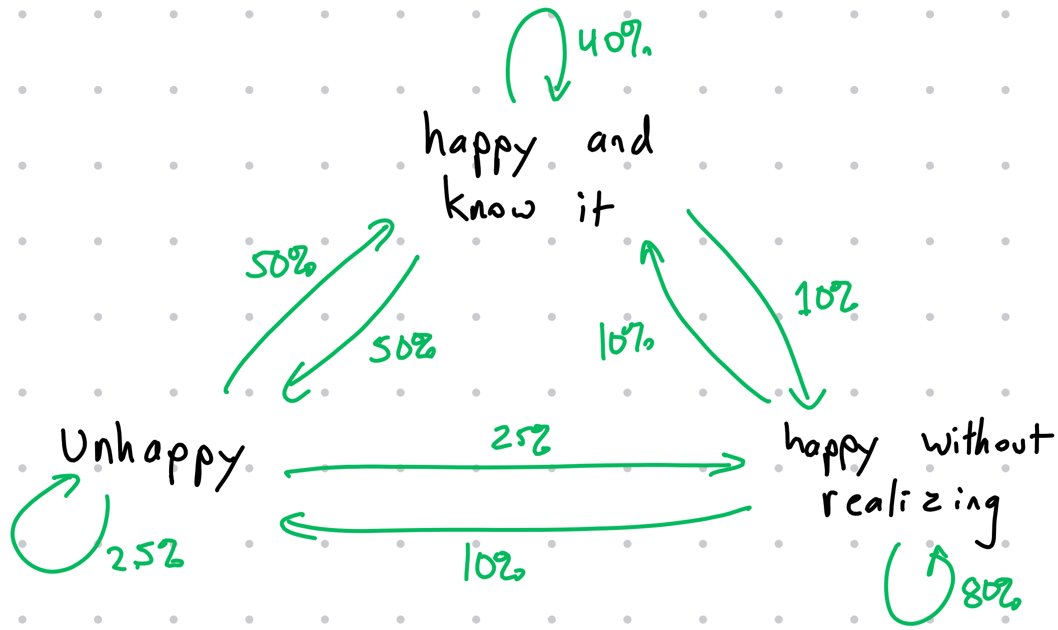
MATH 220

14 March 2025



NOTATION: u_i = percentage of people unhappy in week i

h_i = " " " happy and know it in week i

r_i = " " " happy without realizing " "

START. week 0: $u_0 = 0.3$, $h_0 = 0.5$, $r_0 = 0.2$

30% 50% 20%

Finding the steady-state vector:

Option 1: take $A^{\text{big power}} \cdot \vec{x}_0$

Option 2: Solve $A\vec{x} = \vec{x}$

$$A\vec{x} - I\vec{x} = \vec{0}$$

$$\underbrace{(A-I)}_{\text{matrix}}\vec{x} = \vec{0}$$

Then rescale the solution to make it a probability vector (entries sum to 1)

solution:

$u = 0.55r$

$h = 0.625r$

r is free

Choose r to be anything, such as $r = 1$:

$$\begin{bmatrix} 0.55 \\ 0.625 \\ 1 \end{bmatrix}$$

The entries add up to 2.175, so divide by this to get a probability vector.

$$\begin{bmatrix} 0.252 \\ 0.287 \\ 0.459 \end{bmatrix}$$

Linear Algebra – Day 16

MATH 220

Recall the scenario from the first day of class: Scientists have been watching the spread of happiness this year. Each week, the same 1000 people are checked to find out whether they are unhappy, happy and they know it, or happy without realizing.

- **Of those who are currently unhappy:** Next week, 50% will become happy and know it and 25% will become happy without realizing.
- **Of those currently happy and they know it:** Next week, 50% will become unhappy and 10% will remain happy but not realize it anymore.
- **Of those currently happy without realizing:** Next week, 10% will become unhappy and 10% will now know they are happy.

Define some variables:

- u_i is the *percentage* of people who are unhappy on week i
- h_i is the *percentage* of people who are happy and they know it on week i
- r_i is the *percentage* of people who are happy without realizing it on week i

Suppose that $u_0 = 0.3$, $h_0 = 0.5$, and $r_0 = 0.2$ are known.

1. What are the values of u_1, h_1, and r_1?

$$u_1 = 0.25\underset{u_0}{(0.3)} + 0.5\underset{h_0}{(0.5)} + 0.1\underset{r_0}{(0.2)} = 0.345$$

$$h_1 = 0.5(0.3) + 0.4(0.5) + 0.1(0.2) = 0.37$$

$$r_1 = 0.25(0.3) + 0.1(0.5)\ 0.8(0.2) = 0.285$$

2. Jonah: Wow!

Josie: What now, Jonah?

Jonah: We can write an equation to express u_{i+1} in terms of u_i, h_i, and r_i.

Josie: Yeah! We can also write an equation for h_{i+1} and a third equation for r_{i+1}.

Group discussion: "Finish" the equations by filling in the missing underlined numbers:

$$u_{i+1} = \underline{0.25}\,u_i + \underline{0.5}\,h_i + \underline{0.1}\,r_i$$

$$h_{i+1} = \underline{0.5}\,u_i + \underline{0.4}\,h_i + \underline{0.1}\,r_i$$

$$r_{i+1} = \underline{0.25}\,u_i + \underline{0.1}\,h_i + \underline{0.8}\,r_i$$

unknown

usually

$A\vec{x} = \vec{b}$

unknown

3. Jonah: We have something kind of like a system of equations here.

Group chat: Discuss why Jonah says "something kind of like."

Josie: Yes! Now we can try to use vectors and matrices. I knew linear algebra was cool!

Jonah: Let's put the percentages for each week into a vector: $\mathbf{x}_i = \begin{bmatrix} u_i \\ h_i \\ r_i \end{bmatrix}$.

Group chat: What are $\mathbf{x}_0$ and $\mathbf{x}_1$?

week 0:

$$\vec{x}_0 = \begin{bmatrix} u_0 \\ h_0 \\ r_0 \end{bmatrix}$$

week 1

$$\vec{x}_1 = \begin{bmatrix} u_1 \\ h_1 \\ r_1 \end{bmatrix}$$

4. Jonah: Now we can turn this "system" into a matrix-vector equation $A\mathbf{x}_i = \mathbf{x}_{i+1}$.

Group task: Fill in the numbers in the matrix A:

$$\begin{bmatrix} 0.25 & 0.5 & 0.1 \\ 0.5 & 0.4 & 0.1 \\ 0.25 & 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} u_i \\ h_i \\ r_i \end{bmatrix} = \begin{bmatrix} u_{i+1} \\ h_{i+1} \\ r_{i+1} \end{bmatrix}$$

Stochastic matrix

KADJ: $\vec{x}_1 = A\vec{x}_0$

5. Josie: Oh, so now we have found $\mathbf{x}_1$, we can easily calculate $\mathbf{x}_2$, $\mathbf{x}_3$, and even $\mathbf{x}_{52}$.

Group chat: How would you find $\mathbf{x}_2$? What about $\mathbf{x}_{52}$?

$\vec{x}_2 = A\vec{x}_1 = AA\vec{x}_0 = A^2\vec{x}_0$

$\vec{x}_3 = A^3\vec{x}_0$

$\vec{x}_{52} = A^{52}\vec{x}_0 = \begin{bmatrix} 0.252 \\ 0.287 \\ 0.459 \end{bmatrix}$

6. What percentage of those who are unhappy today will be happy (and they know it) 7 weeks from today?

☞ You'll probably want to pull out Mathematica quickly.

7. How might you find the three percentages "a really long time from now"?

steady-state vector

$A\vec{v} = \vec{v}$

8. Spicy Group Problem: What would the percentages u_0, h_0, r_0 have to be in order for the percentages to *remain exactly the same* next week?

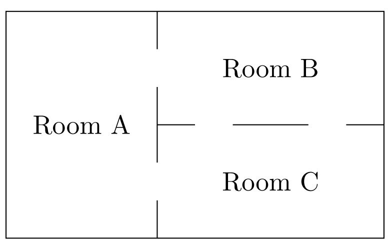
☞ In other words, nothing changes from week to week.

The percentages given in the steady-state vector!

We didn't get to this in class, but here is the solution, as another example.

New scenario: A psychologist places a mouse in a cage with three compartments, as shown in the figure below. The mouse has been trained to select a door *at random* whenever a bell is rung and to move through it into the next compartment.

☞ Each available door is equally likely to be chosen.



1. If the mouse is in Room A:

- Find the probability that the mouse moves to Room B when the bell is rung. — $\frac{1}{2}$
- Find the probability that the mouse moves to Room C when the bell is rung. — $\frac{1}{2}$

2. If the mouse is in Room B:

- Find the probability that the mouse moves to Room A when the bell is rung. — $\frac{1}{3}$
- Find the probability that the mouse moves to Room C when the bell is rung. — $\frac{2}{3}$

3. If the mouse is in Room C:

- Find the probability that the mouse moves to Room A when the bell is rung. — $\frac{1}{3}$
- Find the probability that the mouse moves to Room B when the bell is rung. — $\frac{2}{3}$

In the long run, what proportion of its time will the mouse spend in each room?

☞ Set up some notation for this problem!

See next page!

Stochastic Matrix:

$$M = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{2}{3} & 0 \end{bmatrix}$$

rows represent next room: A, B, or C

columns represent current room: A, B, or C

To find the steady-state vector, solve $(M-I)\vec{x} = \vec{0}$.

$$M - I = \begin{bmatrix} -1 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & -1 & \frac{2}{3} \\ \frac{1}{2} & \frac{2}{3} & -1 \end{bmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{bmatrix} 1 & 0 & -\frac{2}{3} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution: $x_1 = \frac{2}{3}x_3$, $x_2 = x_3$, x_3 is free

Choose a value for x_3. How about: $x_3 = 1$. Then solution is $\vec{x} = \begin{bmatrix} \frac{2}{3} \\ 1 \\ 1 \end{bmatrix}$.

Rescale to get a probability vector: $\frac{2}{3} + 1 + 1 = \frac{8}{3}$

So:

$$\frac{3}{8}\begin{bmatrix} \frac{2}{3} \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{8} \\ \frac{3}{8} \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.375 \\ 0.375 \end{bmatrix}$$

This is the steady-state vector!

The mouse spends $\frac{1}{4}$ of its time in Room A,
$\frac{3}{8}$ of its time in Room B,
and $\frac{3}{8}$ of its time in Room C.