MATH 220

12 March 2025

NOTICE:

identity matrix $I_2$

matrix product $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrices $\overset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}$ and $\overset{A^{-1}}{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}$ are inverses!

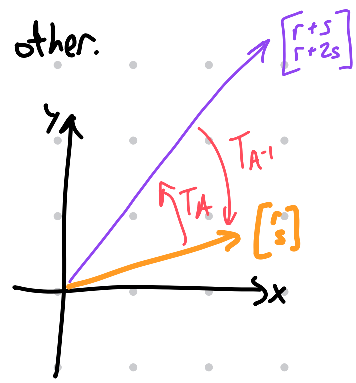
This means: their product (in either order) is the identity.

As linear transformations, they "undo" each other.

$$T_A\left(\begin{bmatrix} r \\ s \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} r+s \\ r+2s \end{bmatrix}$$

$$T_{A^{-1}}\left(\begin{bmatrix} r+s \\ r+2s \end{bmatrix}\right) = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} r+s \\ r+2s \end{bmatrix} = \begin{bmatrix} 2(r+s)-1(r+2s) \\ -1(r+s)+1(r+2s) \end{bmatrix}$$

$$= \begin{bmatrix} r \\ s \end{bmatrix}$$



# Linear Algebra – Day 15

MATH 220

**1. Warm up:** Find the solution(s) to the following:

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad \vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\qquad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 0 \end{array}\right]$

👈 These are just linear systems. How do we solve these, again?

(b) $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\qquad \vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\qquad \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 1 \end{array}\right]$

**2. Simon:** I could have solved (a) and (b) simultaneously by row reducing *one* $2 \times 4$ matrix! Wow!

**Group chat:** What does Simon mean?

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow{-R_2+R_1 \to R_1} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$A \qquad I \qquad\qquad I \qquad A^{-1}$

**3.** Let $\quad A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$. $\qquad A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

Solve the matrix equation $A\mathbf{x} = \mathbf{b}$ by multiplying each side of the equation by $A^{-1}$.

$A\vec{x} = \vec{b}$ is $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \vec{x} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$ so: $\underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}_{A^{-1}} \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\vec{x}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}_{A^{-1}} \underbrace{\begin{bmatrix} 7 \\ 9 \end{bmatrix}}_{\vec{b}}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 14-9 \\ -7+9 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 14-9 \\ -7+9 \end{bmatrix}$

check: $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 5+2 \\ 5+4 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$ ✓ $\qquad \boxed{\vec{x} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}}$

**4. Cleo:** Hey Milo! I want to solve this for the mystery matrix $X$:

$$AX = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad A^{-1}AX = A^{-1}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A^{-1}$$

**Milo:** I get it! Let's try something similar to part (a) and multiply both sides by $A^{-1}$.

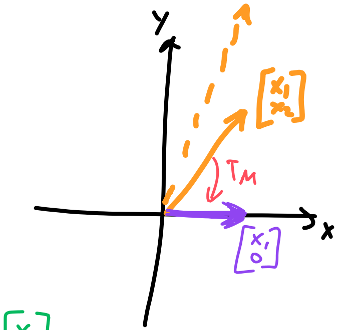
**Cleo:** OK! So when we do that, the $A$ on the left gets canceled and we are left with

$$X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A^{-1}$$

**Milo (shaking head sadly):** Oh, Cleo...

**Group Discussion:** Why is Milo sad?

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A^{-1}$ is not the same as $A^{-1} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$




**5.** Suppose $M = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$ and $N = \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$.

(a) Is matrix $M$ *invertible*? In other words, does $M^{-1}$ exist? Why or why not?

$$\left[\begin{array}{cc|cc}1 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{array}\right]$$

$$T_M\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_1\\0\end{bmatrix}$$

Conclude: $M^{-1}$ does not exist

(b) How about matrix $N$? Is it invertible? Why or why not?

$$\left[\begin{array}{cc|cc}1 & 2 & 1 & 0\\2 & 4 & 0 & 1\end{array}\right] \to \left[\begin{array}{cc|cc}1 & 2 & 1 & 0\\0 & 0 & -2 & 1\end{array}\right]$$

$N^{-1}$ does not exist

**6.** Suppose $C = \begin{bmatrix}1 & 5\\3 & 12\end{bmatrix}$. Find $C^{-1}$ by reducing the "augmented" matrix $[\ C \mid I\ ]$.

$$\left[\begin{array}{cc|cc}1 & 5 & 1 & 0\\3 & 12 & 0 & 1\end{array}\right] \longrightarrow \left[\begin{array}{cc|cc}1 & 5 & 1 & 0\\0 & -3 & -3 & 1\end{array}\right] \to \left[\begin{array}{cc|cc}1 & 5 & 1 & 0\\0 & 1 & 1 & -\frac{1}{3}\end{array}\right]$$

$$\to \left[\begin{array}{cc|cc}1 & 0 & -4 & \frac{5}{3}\\0 & 1 & 1 & -\frac{1}{3}\end{array}\right] \qquad \boxed{C^{-1} = \begin{bmatrix}-4 & \frac{5}{3}\\1 & -\frac{1}{3}\end{bmatrix}}$$

Class ended here.

Here are the solutions to #7-9, for reference

**7.** Suppose $D = \begin{bmatrix}1 & 3 & 2\\-1 & -7 & 6\\0 & 4 & -8\end{bmatrix}$. Try reducing the "augmented" matrix $[\ D \mid I\ ]$. What happens? What does this tell you about matrix $D$?

$$\left[\begin{array}{ccc|ccc}1 & 3 & 2 & 1 & 0 & 0\\-1 & -7 & 6 & 0 & 1 & 0\\0 & 4 & -8 & 0 & 0 & 1\end{array}\right] \to \left[\begin{array}{ccc|ccc}1 & 3 & 2 & 1 & 0 & 0\\0 & -4 & 8 & 1 & 1 & 0\\0 & 4 & -8 & 0 & 0 & 1\end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc}1 & 3 & 2 & 1 & 0 & 0\\0 & -4 & 8 & 1 & 1 & 0\\0 & 0 & 0 & 0 & 0 & 0\end{array}\right]$$

We cannot reduce this side to $I_3$.
Thus, matrix D is not invertible!

**8.** Let $A = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}$ and $B = \begin{bmatrix}2 & 1\\5 & 3\end{bmatrix}$.

(a) Compute $(AB)^{-1}$.

$$AB = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\5 & 3\end{bmatrix} = \begin{bmatrix}7 & 4\\12 & 7\end{bmatrix}, \text{ and } (AB)^{-1} = \begin{bmatrix}7 & 4\\12 & 7\end{bmatrix}^{-1} = \begin{bmatrix}7 & -4\\-12 & 7\end{bmatrix}$$

☞ first compute $AB$, and then find the inverse of the product

same!

(b) Compute $A^{-1}B^{-1}$ and also $B^{-1}A^{-1}$.

$$A^{-1}B^{-1} = \begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}3 & -1\\-5 & 2\end{bmatrix} = \begin{bmatrix}11 & -4\\-8 & 4\end{bmatrix} \neq (AB)^{-1} \quad \text{and} \quad B^{-1}A^{-1} = \begin{bmatrix}3 & -1\\-5 & 2\end{bmatrix}\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix} = \begin{bmatrix}7 & -4\\-12 & 7\end{bmatrix} = (AB)^{-1}$$

☞ first compute $A^{-1}$ and $B^{-1}$ separately, and then multiply the two inverses.

**9.** (Challenge) Suppose that $A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$. Find $A^{-1}$ by reducing the "augmented" matrix $[\ A \mid I\ ]$ by hand.

☞ VARIABLES! YAY! Do your thing, don't let them scare you.

$$\left[\begin{array}{cc|cc}a & b & 1 & 0\\c & d & 0 & 1\end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|cc}1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc}\\0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc}\end{array}\right]$$