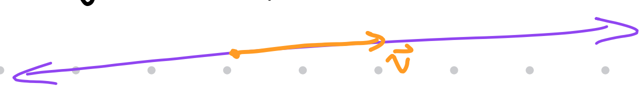
MATH 220

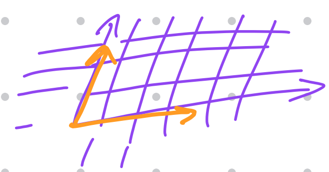
10 March 2025

Comment about solution sets and geometry

- If a solution set has exactly <u>one</u> free variable, then it is geometrically a <u>line</u>.



- If a solution set has exactly <u>two</u> free variables, then it is a plane



- More than two free variables: hyperplane
- No free variables: a single point

COMPOSITIONS OF LINEAR TRANSFORMATIONS

Let f and g be functions.

$(f \circ g)(x) = f(g(x))$ is the <u>composition</u> of f and g

means: do g, then do f

Example: $f(x) = \sqrt{x}$ and $g(x) = 3x - 1$,

then $(f \circ g)(x) = f(g(x)) = f(3x-1) = \sqrt{3x-1}$

$T_A: \mathbb{R}^3 \to \mathbb{R}^2$
$T_B: \mathbb{R}^4 \to \mathbb{R}^3$

Linear Algebra – Day 14

MATH 220

1. Suppose $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & -2 & 3 \end{bmatrix}$.

Let T_A and T_B be the linear transformation given by multiplication by A and B respectively.

(a) Does $T_B \circ T_A$ even make sense? NO

If yes, the domain of $T_B \circ T_A$ is ______ and the codomain is ______ .

👉 That is, is it possible to perform T_A first, immediately followed by T_B?

(b) Does $T_A \circ T_B$ even make sense? Yes $\mathbb{R}^4 \xrightarrow{T_B} \mathbb{R}^3 \xrightarrow{T_A} \mathbb{R}^2$

$\begin{bmatrix} 0 & 3 & & \\ 1 & 6 & & \end{bmatrix}$

If yes, the domain of $T_A \circ T_B$ is $\mathbb{R}^4$ and the codomain is $\mathbb{R}^2$.

👉 That is, is it possible to perform T_B first, immediately followed by T_A?

(c) You should have concluded that $T_A \circ T_B$ is the one that makes sense. Find the matrix that performs the transformation $T_A \circ T_B$.

👉 Hint: what happens to the "e" vectors?

$$T_A\left(T_B\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right)\right) = T_A\left(\begin{bmatrix} -1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & -2 & 3 \end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right) = T_A\left(\begin{bmatrix}-1\\0\\1\end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$T_A\left(T_B\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right)\right) = T_A\left(\begin{bmatrix}0\\1\\2\end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix}$$

(d) Compute AB. What do you notice?

$$AB = \underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix}}\underset{B}{\begin{bmatrix} -1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & -2 & 3 \end{bmatrix}} = \begin{bmatrix} 0 & 3 & 0 & 3 \\ 1 & 6 & -2 & 7 \end{bmatrix}$$

2. Felix: I need to multiply $\begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ times $\begin{bmatrix} 3 & -1 & 5 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. upper triangular matrices

Ava: You will get a 3×3 matrix with at least ~~six~~ three entries that are 0.

Group chat: What is Ava talking about? Compute the matrix product to see if Ava is correct.

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & 5 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 26 \\ 0 & -2 & 12 \\ 0 & 0 & 4 \end{bmatrix}$$

Product of upper triangular matrices is <u>always</u> upper triangular.

3. Milo: Matrix multiplication is hard!

Chloe: Sometimes it's not! Try multiplying the matrices $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Diagonal matrices!

Group chat: What do you think Chloe means?

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Product of diagonal matrices is diagonal!

Group chat: What other examples of matrices can you think of that are *easy* to multiply?

We didn't get to #4 in class, but here is the solution:

4. Compute the matrix product:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 4 \\ 1 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 4 \\ 7 & 5 & 1 \end{bmatrix}$$

Milo: You just did an elementary row operation!

Group chat: What is Milo talking about?

Let $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 4 \\ 1 & 5 & 4 \end{bmatrix}$. The product EA is the same as applying the elementary row operation $3R_1 + R_3 \to R_3$ to A.

Chloe: Did you know that *every* elementary row operation is really matrix multiplication in disguise?

Group chat: What is Chloe talking about? Can you explain how each of the three types of elementary row operations can be performed by multiplying matrices?

To perform an elementary row operation on matrix A, compute a product EA, where E has a form like one of the following:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ k & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

multiplies a single row by k

swaps two rows

adds k times one row to another row

5. Can you find each of the mystery matrices?

Feel free to try #5. We will return to this idea next time!

(a) $\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 5 \\ 7 & 9 \end{bmatrix} \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$