MATH 220
7 March 2025

EXAMPLE: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ let $T_A(\vec{x}) = A\vec{x}$ $(\mathbb{R}^3 \to \mathbb{R}^3)$

- Is T_A onto? Does $A\vec{x} = \vec{b}$ have a solution for all $\vec{b} \in \mathbb{R}^3$?

Row Reduce: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 - x_3 = 0$
$x_2 + 2x_3 = 0$
$\underbrace{x_3 \text{ is free}}_{A\vec{x} = \vec{0}}$

$-1\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$ ⟸ columns of A are linearly dependent

Two pivots in 3 rows
means that $A\vec{x} = \vec{b}$ has no solution for some $\vec{b} \in \mathbb{R}^3$.
So, A is not onto.

- Is T_A one-to-one? Does $A\vec{x} = \vec{b}$ have a unique solution for all $\vec{b} \in \mathbb{R}^3$?

It suffices to consider $A\vec{x} = \vec{0}$.

If $A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{b}$ for $\vec{x} \neq \vec{y}$, then

$A\vec{x} - A\vec{y} = \vec{b} - \vec{b}$

$A(\vec{x} - \vec{y}) = \vec{0}$ ⟵ nontrivial solution to $A\vec{x} = \vec{0}$.

→ Does $A\vec{x} = \vec{0}$ have a nontrivial solution?

Yes, because there is not a pivot in each column of the RREF of A.

So, T_A is not one-to-one.

Linear Algebra – Day 13

MATH 220

1. The 4 matrices below are associated with the linear transformations, T_A, T_B, T_C, and T_D.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

pivot in every row $\iff$ onto

(a) Determine which of the linear transformations is *onto* by determining if every vector in the codomain of the transformation is in the span of the columns of the corresponding matrix.

☞ RREF! Use Mathematica!

$B \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ T_B is onto $\quad C \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ T_C is onto $\quad D \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ T_D is not onto

(b) Determine which of the linear transformations is one-to-one by determining how many pre-images of $\mathbf{0}$ there are.

pivot in every column $\iff$ one-to-one

T_B is one-to-one, T_C is not one-to-one, T_D is not one-to-one

2. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ with $m < n$, so the corresponding matrix has fewer rows than columns. Is the transformation *always*, *sometimes*, or *never* one-to-one? Explain.

☞ Look at T_C

more columns than rows $\Longrightarrow$ not a pivot in every column

never one-to-one

3. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ with $m > n$, so the corresponding matrix has more rows than columns. Is the transformation *always*, *sometimes*, or *never* onto? Explain.

☞ Look at T_D

never onto

We skipped this in class, but here are the solutions.

4. Consider the transformation T_A above.

(a) Describe the set of pre-images of $\mathbf{0}$ in vector form. (Remember that this is the set of solutions to $A\mathbf{x} = \mathbf{0}$.)

$A\vec{x} = \vec{0}$ has solution $x_1 = x_3$, $x_2 = -2x_3$, and x_3 is free.

In vector form, the solutions are $x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ for all $x_3 \in \mathbb{R}$.

(b) Describe the set of pre-images of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in vector form (i.e., the set of solutions to $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$).

$$\left[\begin{array}{c|c} A & \begin{matrix} 1 \\ 1 \\ 1 \end{matrix} \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so $x_1 = -1 + x_3$
$x_2 = 1 - 2x_3$
x_3 free

Vector form of solution set:

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

for all $x_3 \in \mathbb{R}$

(c) What is the geometric relationship between the two solution sets above?

The two solution sets are two parallel lines in $\mathbb{R}^3$.

5. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ -2 & 0 \\ -1 & 3 \end{bmatrix}$, and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$.

(a) Which of the sums $\underbrace{A+B}_{\text{yes}}$, $A+C$, and $B+C$ can you compute?

yes
Same size

No, not the same size!

(b) Compute $A+B$, $2A$, and $C+D^T$.

$$A+B = \begin{bmatrix} 1 & 3 & 2 \\ 6 & 6 & 6 \\ 8 & 7 & 10 \end{bmatrix}, \qquad 2A = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix} = A+A,$$

$$C+D^T = \begin{bmatrix} 1 & 2 \\ -2 & 0 \\ -1 & 3 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}}_{D^T} = \begin{bmatrix} 2 & 2 \\ -2 & 2 \\ -1 & 3 \end{bmatrix}$$

(c) Which of the products $\underbrace{AB}_{\text{yes}}$, AC, AD, CA, CB and CD can you compute?

(d) Compute AB by hand, and BA using Mathematica — use `B.A` rather than `B*A`. In fact, what does `B*A` give you?

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 2 \\ 16 & 3 & 2 \\ 25 & 6 & 2 \end{bmatrix}$$

We will return to this on Monday.

6. Suppose $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & -2 & 3 \end{bmatrix}$.

Let T_A and T_B be the linear transformation given by multiplication by A and B respectively.

(a) Does $T_B \circ T_A$ even make sense?

If yes, the domain of $T_B \circ T_A$ is ______ and the codomain is ______ .

☞ That is, is it possible to perform T_A first, immediately followed by T_B?

(b) Does $T_A \circ T_B$ even make sense?

If yes, the domain of $T_A \circ T_B$ is ______ and the codomain is ______ .

☞ That is, is it possible to perform T_B first, immediately followed by T_A?

(c) You should have concluded that $T_A \circ T_B$ is the one that makes sense. Find the matrix that performs the transformation $T_A \circ T_B$.

☞ Hint: what happens to the "e" vectors?

(d) Compute AB. What do you notice?