Linear Algebra – Day 11

MATH 220

1. *Re-enact the following dialogue with your group.*

Milo: Hey, Maura! I have this function $T : \mathbb{R}^3 \to \mathbb{R}^2$ that has the formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 2 \\ x_2 + x_3 \end{bmatrix}.$$

Maura: That function is *not* a linear transformation!

Milo: But why? I looks like a wonderful function.

Maura: We can find an example of vectors $\mathbf{u}$ and $\mathbf{v}$ where $T(\mathbf{u}+\mathbf{v})$ does not equal $T(\mathbf{u})+T(\mathbf{v})$.

Group discussion: Try to find two specific vectors $\mathbf{u}$ and $\mathbf{v}$ such that $T(\mathbf{u}+\mathbf{v}) \neq T(\mathbf{u})+T(\mathbf{v})$.

Milo: This function actually fails *both* requirements of a linear transformation! I can find an example of a vector $\mathbf{u}$ and a scalar c where $T(c \cdot \mathbf{u})$ does not equal $c \cdot T(\mathbf{u})$.

Group discussion: Try to find a specific vector $\mathbf{u}$ and scalar c such that $T(c \cdot \mathbf{u}) \neq c \cdot T(\mathbf{u})$.

2. Now consider at the function $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by the formula $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ x_2 + x_3 \end{bmatrix}.$

(a) Use the formula above to calculate $T\left(\begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}\right).$

(b) Use the formula above to individually calculate $T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) + T\left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}\right).$

(c) Are the results of parts (a) and (b) equal?

(d) Use the formula above to calculate $T\left(\begin{bmatrix} c \cdot a_1 \\ c \cdot a_2 \\ c \cdot a_3 \end{bmatrix}\right).$

(e) Use the formula above to calculate $c \cdot T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right).$

(f) Are the results of parts (d) and (e) equal?

3. Here is the formula for a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2y \\ 3x + 7y \\ -y \end{bmatrix}.$$

👉 I already checked to make sure T follows the two requirements!

(a) Using the formula for T above, calculate $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$, $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$, and $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$.

(b) **Ava:** I figured out that $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}$.

👉 REMEMBER: linear transformations don't mess with linear combinations.

Jason: Well done, Ava! That is correct.

Ava: But I don't need the formula to figure out $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$.

Jason: What do you mean? Are you a magician?

Ava: No, silly! I just used the work I already did and got

$$T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = 2 \cdot \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}.$$

Group chat: what did Ava do to figure out $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$?

(c) Try to come up with a 3×2 matrix A for which $T(\mathbf{x}) = A\mathbf{x}$.

4. Let $A = \begin{bmatrix} 2 & 0 & -2 \\ -1 & 3 & 4 \end{bmatrix}$ and let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $T(\mathbf{x}) = A\mathbf{x}$. Is the vector $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$ in the range of T?

5. Suppose $T : \mathbb{R}^4 \to \mathbb{R}^3$ is the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & -1 & 1 & -4 \\ 1 & 0 & 3 & -2 \\ 2 & 2 & 10 & 0 \end{bmatrix}$$

(a) Rephrase the following question in as many ways as you possibly can: *Is every vector* $\mathbf{b}$ *in* $\mathbb{R}^3$ *in the range of* T*?*

(b) What is the answer? Is every vector $\mathbf{b}$ in $\mathbb{R}^3$ in the range of T?